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Riesz Basis in de Branges Spaces of Entire Functions

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Abstract

In this paper we consider the problem of Riesz basis in de Branges spaces of entire functions $\mathcal{H}(E)$ with the condition that $\varphi'(x) \geq \alpha > 0$, where φ is the corresponding phase function. We are concerned with the sets of real numbers $\{\lambda_n\}$ such that the normalized reproducing kernels $k(\lambda_n, .)/\|k(\lambda_n, .)\|$ satisfies the restricted isometry property, which in turn constitute a Riesz basis in $\mathcal{H}(E)$. Then we give a criterion on stability of reproducing kernels corresponding to real points which form a Riesz basis in $\mathcal{H}(E)$ with respect to small perturbations, which generalize some well-known Riesz basis perturbation results in the Paley-Wiener space.

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1 Introduction

Compressive sensing provides an alternative method for efficiently acquiring and reconstructing a signal to the Shannon sampling theorem when the signal under acquisition is known to be sparse or compressible. Recently, Candès and Tao [4] introduced very intense activity related to compressed sensing, known as the restricted isometry property, which is also known as the uniform uncertainty principle. The restricted isometry property generalizes the notion of coherence, and allow recovering and extending many known compressive sampling results.

In this paper we work in the context of a reproducing kernel Hilbert spaces. In these spaces the restricted isometry property is a very convenient tool which allows one to reconstruct a signal from its sampling values. It is known that a frame which satisfies a restricted isometry property with isometry constant $\delta < 1$ act as an orthogonal basis. For this reason, one of the main interests of the present paper is to understand what properties of a sequence $\{\lambda_n\}$ of real numbers guarantee that the corresponding normalized reproducing kernels

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satisfies a restricted isometry property in de Branges spaces $\mathcal{H}(E)$ of entire functions as a special class of reproducing kernel Hilbert spaces. Theory of de Branges spaces is an important branch of modern analysis having numerous interesting applications in mathematical physics, harmonic analysis and even number theory.

The problem of description of Riesz bases of normalized reproducing kernels is one of intriguing open problems in the area, results in this direction would be of interests for specialists in de Branges theory and its applications. In spite of many deep and important results, there is still no explicit description of bases in general de Branges spaces. The present paper studies stability of Riesz bases of reproducing kernels in the class of de Branges spaces with the condition that $\varphi'(x) \geq \alpha > 0$ on $\mathbb{R}$, where φ is an important characteristic of a de Branges space known as a phase function. Specifically, we are concerned with the sets of real numbers $\Lambda = \{\lambda_n\}$ such that the normalized reproducing kernels $k(\lambda_n, .)/\|k(\lambda_n, .)\|$ constitute a Riesz basis. We also prove new results on stability of reproducing kernels corresponding to real points which form a Riesz basis in $\mathcal{H}(E)$ with respect to small perturbations, which generalize some well-known Riesz basis perturbation results in the Paley-Wiener space.

In order to properly state our results, we need to review the main concepts and terminology of the theory of de Branges spaces of entire functions introduced by L. de Branges [13] in connection with inverse spectral problems for differential operators. These spaces generalize the classical Paley-Wiener space which consists of the entire functions of exponential type and square integrable on the real line. More information about these spaces can be found in [8–11].

2 Theory of de Branges spaces

In this section, we present a brief review and some relevant results on de Branges spaces theory. Assume f is an analytic function on the upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$, then f is said to be of *bounded type* in $\mathbb{C}^+$ if it can be written as a quotient of two bounded analytic functions in $\mathbb{C}^+$. The *mean type* of f in $\mathbb{C}^+$ is defined by

$$\mathrm{mt}_+(f) := \limsup_{y\to+\infty} \frac{\log|f(iy)|}{y}.$$

For an entire function f, we define the function f^* as $f^*(z) := \overline{f(\bar{z})}$. The *Hermite-Biehler* class, denoted by $\mathcal{HB}$, consists of all entire functions $E(z)$ that has no zeros in the upper half-plane and satisfies the condition

$$|E(\bar{z})| < |E(z)|, \quad \text{whenever } \Im z > 0. \tag{1}$$

Given a function $E \in \mathcal{HB}$, the associated de Branges space $\mathcal{H}(E)$ consists of all entire functions $f(z)$ such that

$$||f||_E^2 := \int_{\mathbb{R}} \left|\frac{f(t)}{E(t)}\right|^2 dt < \infty, \tag{2}$$

and $f(z)/E(z)$ and $f^*(z)/E(z)$ are of bounded type and nonpositive mean type in the upper half-plane. This is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_E = \int_{\mathbb{R}} \frac{f(t)\overline{g(t)}}{|E(t)|^2}\, dt.$$

The Hilbert space $\mathcal{H}(E)$ has the special property that, for every nonreal number w, the linear functional defined on the space by $f \mapsto f(w)$ is continuous. Therefore, for every nonreal $w \in \mathbb{C}$ there exists a function $k(w, z)$ in $\mathcal{H}(E)$ such that

$$f(w) = \langle f(t), k(w, t)\rangle_E, \tag{3}$$

for every $f \in \mathcal{H}(E)$. Property (3) is known as the *reproducing kernel property*. The function $k(w, z)$ is called the *reproducing kernel* of $\mathcal{H}(E)$, which is given by (see [13, Theorem 19])

$$k(w, z) = \frac{\bar{E}(w)E(z) - E(\bar{w})E^*(z)}{2\pi i(\bar{w} - z)}. \tag{4}$$

An important feature of the de Branges space $\mathcal{H}(E)$ is the phase function corresponding to the generating function E, that is, for any entire function $E \in \mathcal{HB}$, there exists a continuous and strictly increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $E(x)e^{i\varphi(x)} \in \mathbb{R}$ for all $x \in \mathbb{R}$, essentially, $\varphi = -\arg(E)$ on $\mathbb{R}$, and $E(x)$ can be written as

$$E(x) = |E(x)|e^{-i\varphi(x)}, \quad x \in \mathbb{R}. \tag{5}$$

If a function φ has these properties then it is referred to as a *phase function* of E. It follows that a phase function of E is defined uniquely up to an additive constant, a multiple of 2π. If $\varphi(x)$ is any such function, and $E(x) \neq 0$, then using (4) and (5), an easy computation gives

$$\|k(x, .)\|^2 = k(x, x) = \frac{1}{\pi}\varphi'(x)|E(x)|^2. \tag{6}$$

The leading example of de Branges spaces is the Paley-Wiener space

$$\mathcal{H}(e^{-i\pi z}) = \mathcal{P}\mathrm{W}_\pi,$$

consists of square-integrable functions on the real line whose Fourier transforms are supported on $[-\pi, \pi]$. The reproducing kernel for $\mathcal{P}\mathrm{W}_\pi$ is $k(w, z) = \frac{\sin \pi(z-\bar{w})}{\pi(z-\bar{w})}$, $w, z \in \mathbb{C}$, $z \neq \bar{w}$, and the corresponding phase function $\varphi(x) = \pi x$.

A key feature of a de Branges space is that it always has a basis consisting of reproducing kernels corresponding to real points, [2].

Theorem 2.1. *Let $\mathcal{H}(E)$ be a de Branges space and $\varphi(x)$ be a phase function associated with E. If $\alpha \in \mathbb{R}$, and $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ is a sequence of real numbers, such that $\varphi(\lambda_n) = \alpha + \pi n$, $n \in \mathbb{Z}$, then The functions $\{k(\lambda_n, z)\}_{n\in\mathbb{Z}}$ form an orthogonal set in $\mathcal{H}(E)$.*

3

If $e^{i\alpha}E(z)-e^{-i\alpha}E^*(z)\notin\mathcal{H}(E)$, then $\left\{\frac{k(\lambda_n,z)}{\|k(\lambda_n,.)\|}\right\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $\mathcal{H}(E)$. Moreover, for every $f(z)\in\mathcal{H}(E)$,

$$f(z)=\sum_{n\in\mathbb{Z}} f(\lambda_n)\frac{k(\lambda_n,z)}{\|k(\lambda_n,.)\|^2}\,, \tag{7}$$

and

$$\|f\|^2=\sum_{n\in\mathbb{Z}}\left|\frac{f(\lambda_n)}{E(\lambda_n)}\right|^2\frac{\pi}{\varphi'(\lambda_n)}. \tag{8}$$

A central tool in our proofs is the following Bernstein inequality in de Branges spaces introduced by A. Baranov, whose proof can be found in [2]:

Lemma 2.2. *Let $E\in\mathcal{HB}$ be such that $E'/E\in\mathbb{H}^\infty(\mathbb{C}^+)$, then*

$$\|f'/E\|_2\le C_{Ber}\|f\|_E$$

for all $f\in\mathcal{H}(E)$, where $C_{Ber}=(4+\sqrt{6})\|E'/E\|_\infty$.

3 Basis Theory

In this section we recall some basic concept of frames and Riesz bases for Hilbert spaces (see for example, Daubechies [7]; Duffin and Schaeffer [14]).

A family of elements $\{f_n\}_{n=1}^\infty$ in a separable Hilbert space $\mathcal{H}$ forms a frame if there exist $0<A\le B<\infty$ such that

$$A\|f\|^2\le\sum_{n=1}^\infty|\langle f,f_n\rangle|^2\le B\|f\|^2,\quad\text{for all }\ f\in\mathcal{H}. \tag{9}$$

The constants A,B in (9) are called the *frame bounds* for $\{f_n\}_{n=1}^\infty$. If the two frame bounds are equal we call a frame $\{f_n\}_{n=1}^\infty$ a *tight frame*. For each $f\in\mathcal{H}$ we have the *frame expansions*

$$f=\sum_{n=1}^\infty\langle f,f_n\rangle\tilde{f}_n=\sum_{n=1}^\infty\langle f,\tilde{f}_n\rangle f_n, \tag{10}$$

with unconditional convergence of these series, where $\{\tilde{f}_n\}$ is the dual frame of $\{f_n\}$. If, in addition to (9), $\{f_n\}_{n=1}^\infty$ is a linearly independent set, we call it a *Riesz basis* for $\mathcal{H}$. An equivalent characterization for a sequence $\{f_n\}_{n=1}^\infty$ to be a Riesz basis is that $\{f_n\}_{n=1}^\infty$ be a complete sequence in $\mathcal{H}$ and there exist positive constants A and B such that

$$A\sum_n|c_n|^2\le\left\|\sum_n c_nf_n\right\|_{\mathcal{H}}^2\le B\sum_n|c_n|^2\,, \tag{11}$$

for all finite sequences of scalars $\{c_n\}$, see [20].

If the Reisz basis is an orthogonal basis, then $A = B = 1$. Hence, a Riesz basis is automatically a frame, moreover, inequality in (9) holds with the same constants A and B as the inequality in (11). A Riesz basis $\{f_n\}_{n=1}^{\infty}$ is equivalent to an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for $\mathcal{H}$, namely, if there is a bounded invertible operator $U : \mathcal{H} \to \mathcal{H}$ such that $Uf_n = e_n$. Consequently, any Riesz basis of $\mathcal{H}$ is an unconditional basis of $\mathcal{H}$ but not conversely in general. Because of this parallelism, the Riesz bases is the appropriate framework from which to obtain nonorthogonal sampling formulas. It follows that every $f \in \mathcal{H}$ has a unique expression

$$f = \sum_n \langle f, \tilde{f}_n \rangle f_n$$

where $\tilde{f}_n = U^* U f_n$ are the elements of the dual basis of $\{f_n\}$.

If $\mathcal{H}$ is a reproducing kernel Hilbert space, a sequence $\Lambda = \{\lambda_n\}$ is *interpolating* for $\mathcal{H}$ if there exists an $f \in \mathcal{H}$ satisfying $f(\lambda_n) = a_n$ for any choice of interpolation data $\{a_n / \|k(\lambda_n, .)\|\} \in \ell^2(\mathbb{C})$. It is *complete interpolating* if in addition f is unique. From an equivalent point of view, it is well known that a sequence Λ is an *interpolating sequence* in $\mathcal{H}$ if and only if $\{k(\lambda_n, .)/\|k(\lambda_n, .)\|\}$ is a Riesz sequence, and Λ is a *complete interpolating sequence* if and only if $\{k(\lambda_n, .)/\|k(\lambda_n, .)\|\}$ is a Riesz basis in $\mathcal{H}$, see [17] for more details and discussions.

Definition 3.1. *A sequence $\{f_n\}_{n=1}^{\infty}$ is said to have the* restricted isometry property *if there exists $\delta \in (0, 1)$ such that*

$$(1-\delta) \sum_{n=1}^{\infty} |c_n|^2 \leq \left\| \sum_{n=1}^{\infty} c_n f_n \right\|^2 \leq (1+\delta) \sum_{n=1}^{\infty} |c_n|^2, \tag{12}$$

for any sequence of scalars $\{c_n\}$, where δ is known as the isometry constant.

Although the restricted isometry property is difficult to verify, small restricted isometry constants are desired; the closed δ to zero, the closer to orthogonal basis. On the other hand, this definition in particular means that $\{f_n\}$ is a Riesz basis for its linear span. Conversely, if $\{f_n\}$ is a Riesz basis satisfying (11) then the scaled sequence $\{\sqrt{\frac{2}{B+A}} f_n\}$ satisfies (12) with $\delta = \frac{B-A}{B+A}$. In this work, we approach the problem of stability of Riesz basis of a Hilbert space $\mathcal{H}$. Specifically, given a family $\{g_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ which is close, in some sense, to the Riesz basis (or a frame) $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$, we find conditions to ensure that $\{g_n\}_{n=1}^{\infty}$ is also a Riesz basis (or a frame). This problem is important in practice, and has been studied widely by many authors in the context of bases of exponentials in L^2 on some interval. The first result due to Paley and N. Wiener [18] states that if $\{\lambda_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{R}$ and $\sup_{n\in\mathbb{Z}} |\lambda_n - n| \leq \delta < \frac{1}{\pi^2}$, then the set $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis for the Paley-Wiener space $\mathcal{PW}_\pi$ (in this cae $f_n = e^{inx}$ and $g_n = e^{i\lambda_n x}$). In [19] M. Kadec proved that the result is true for $\delta < \frac{1}{4}$, whereas the conclusion may fail if $\sup_{n\in\mathbb{Z}} |\lambda_n - n| = \frac{1}{4}$ (see [5]). Recently, some results obtained in [3] on the stability of bases and frames of reproducing kernels based on the estimates of derivatives in terms of Carleson measure in model spaces

$K^2_\Theta = \mathbb{H}^2 \ominus \Theta\mathbb{H}^2$ of the Hardy class $\mathbb{H}^2$ in the upper half plane $\mathbb{C}^+$, where Θ is an inner function in $\mathbb{C}^+$.

In the present paper we are particularly interested in the reproducing kernel Hilbert space $\mathcal{H}(E)$, we shall take for the f_n's the normalized reproducing kernel functions $\frac{k(\lambda_n,.)}{\|k(\lambda_n,.)\|}$, where $\Lambda = \{\lambda_n\}$ is a sequence of real numbers. To be exact, we are interested in stability of the basis $\frac{k(\lambda_n,.)}{\|k(\lambda_n,.)\|}$: given a Riesz basis $\frac{k(\lambda_n,.)}{\|k(\lambda_n,.)\|}$ for $\mathcal{H}(E)$ and a set of points μ_n which, in some sense, close to λ_n, whether the system $\frac{k(\mu_n,.)}{\|k(\mu_n,.)\|}$ is also a Riesz basis for $\mathcal{H}(E)$, which, as a result, leads to a Riesz basis expansion.

We will need below the following lemma which will play the key role in our proofs, see Corollary 15.1.5 in [6].

Lemma 3.1. *Let $\{f_n\}_{n=1}^\infty$ be a frame for a Hilbert space $\mathcal{H}$ with bounds A, B, and let $\{g_n\}_{n=1}^\infty$ be a sequence in $\mathcal{H}$. If there exists a constant $R < A$ such that*

$$\sum_{n=1}^{\infty} |\langle f, f_n - g_n\rangle_{\mathcal{H}}|^2 \leqslant R\,\|f\|^2_{\mathcal{H}}, \quad \forall f \in \mathcal{H},$$

then $\{g_n\}_{n=1}^\infty$ is a frame for $\mathcal{H}$ with bounds

$$A(1-\sqrt{R/A}\,)^2, B(1+\sqrt{R/B}\,)^2.$$

If $\{f_n\}_{n=1}^\infty$ is a Riesz basis, then $\{g_n\}_{n=1}^\infty$ is a Riesz basis.

4 Riesz Basis in de Branges Spaces

Given a de Branges space $\mathcal{H}(E)$ with reproducing kernel $k(w,z)$, we can assume, without loss of generality, that E has no real zeros (see [16]), hence $k(x,x) > 0$ for all $x \in \mathbb{R}$ by (6). Let $\Lambda = \{\lambda_n\}_{n=1}^\infty$ be a sequence of real numbers, from now on, we set

$$f_n(z) := \frac{k(\lambda_n, z)}{\|k(\lambda_n, .)\|}, n \in \mathbb{N}, z \in \mathbb{C}. \tag{13}$$

Definition 4.1. *Let $\Lambda = \{\lambda_n\}_{n=1}^\infty$ be a sequence of distinct points. We say that Λ is* sequentially separated *if $|\lambda_{n+1} - \lambda_n| \geq \sigma_n$, for all $n \geq 1$, and $\sigma_n \leq \sigma_{n+1}$ for all $n \geq 1$.*

Next we derive an estimate of the isometry constant δ. This estimate leads to a sufficient condition for a sequence $\{f_n\}$ to have the Restricted Isometry Property.

Lemma 4.1. *Given a de Branges space $\mathcal{H}(E)$, and $\varphi(x)$ a phase function associated with E such that $\varphi'(x) \geq \alpha > 0$ on $\mathbb{R}$. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequentially*

separated sequence of real numbers such that $\sigma_n \geq 1$. *If* $\sum_{n=1}^{\infty} \frac{1}{\sigma_n^2} < \frac{3\alpha^2}{\pi^2}$, *then*

$$\delta := \left(\sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} |\langle f_n, f_m \rangle|^2 \right)^{\frac{1}{2}} < 1 \tag{14}$$

Proof. For any real number x, $E(x) = e^{-i\varphi(x)}|E(x)|$, which implies that $\frac{E(x)}{\overline{E}(x)} = e^{-2i\varphi(x)}$. Let $a, b \in \mathbb{R}$, then using (4) and the fact that $k(a,b) = \langle k(a,.), k(b,.)\rangle$ we get,

$$\begin{aligned} \frac{k(a,b)}{\overline{E}(a)} &= \frac{1}{\overline{E}(a)} \frac{\overline{E}(a)E(b) - E(a)\overline{E}(b)}{2\pi i(a-b)} \\ &= \frac{E(b) - e^{-2i\varphi(a)}\overline{E}(b)}{2\pi i(a-b)}. \end{aligned}$$

Simple calculations then shows that

$$\begin{aligned} \langle \frac{k(a,.)}{\overline{E}(a)}, \frac{k(b,.)}{\overline{E}(b)} \rangle &= \frac{1}{E(b)} \frac{k(a,b)}{\overline{E}(a)} \\ &= \frac{1 - e^{2i(\varphi(b) - \varphi(a))}}{2\pi i(a-b)} \end{aligned}$$

and,

$$\frac{k^2(a,b)}{|E(a)|^2|E(b)|^2} = \frac{\sin^2(\varphi(a) - \varphi(b))}{\pi^2(a-b)^2}.$$

Consequently, since $k(x,x) = \frac{1}{\pi}\varphi'(x)|E(x)|^2$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned} \frac{k^2(a,b)}{k(a,a)k(b,b)} &= \pi^2 \frac{k^2(a,b)}{\varphi'(a)\varphi'(b)|E(a)|^2|E(b)|^2} \\ &= \frac{1}{\varphi'(a)\varphi'(b)} \frac{\sin^2(\varphi(a) - \varphi(b))}{(a-b)^2} \end{aligned}$$

In particular, for f_n defined in (13) we have

$$\begin{aligned} |\langle f_n, f_m \rangle|^2 &= \left| \langle \frac{k(\lambda_n,.)}{\|k(\lambda_n,.)\|}, \frac{k(\lambda_m,.)}{\|k(\lambda_m,.)\|} \rangle \right|^2 \\ &= \frac{1}{\varphi'(\lambda_m)\varphi'(\lambda_n)} \frac{\sin^2(\varphi(\lambda_m) - \varphi(\lambda_n))}{(\lambda_m - \lambda_n)^2} \\ &\leq \frac{1}{\alpha^2} \frac{1}{(\lambda_m - \lambda_n)^2} \end{aligned}$$

because $\varphi'(x) \geq \alpha$ on $\mathbb{R}$ by the hypothesis. Since $\{\lambda_n\}$ is sequentially separated and $\sigma_n \geq 1$ then for $m > n$, $m = n + k$, for some $k \geq 1$, and

$$(\lambda_m - \lambda_n) \geq (m-n)\sigma_n = k\sigma_n$$

Therefore, for any $n \geq 1$,

$$\begin{aligned}\sum_{m=n+1}^{\infty} |\langle f_n, f_m\rangle|^2 &\leq \frac{1}{\alpha^2} \sum_{m=n+1}^{\infty} \frac{1}{(\lambda_m - \lambda_n)^2} \\ &\leq \frac{1}{\alpha^2} \sum_{m=n+1}^{\infty} \frac{1}{(m-n)^2 \sigma_n^2} \\ &\leq \frac{1}{\alpha^2 \sigma_n^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \frac{1}{\alpha^2 \sigma_n^2}.\end{aligned}$$

Consequently,

$$\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2 = 2\sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} |\langle f_n, f_m\rangle|^2 \leq \frac{\pi^2}{3\alpha^2} \sum_{n=1}^{\infty} \frac{1}{\sigma_n^2}.$$

From this the conclusion follows with $\delta < 1$. □

Next we apply the estimate obtained in Lemma 4.1 to give conditions for the sequence $\{f_n\}$ to have the Restricted Isometry Property.

Theorem 4.2. *Given a de Branges space $\mathcal{H}(E)$, and $\varphi(x)$ a phase function associated with E such that $\varphi'(x) \geq \alpha > 0$ on $\mathbb{R}$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequentially separated sequence of real numbers such that $\sigma_n \geq 1$, $\forall n \geq 1$. If $\sum_{n=1}^{\infty} \frac{1}{\sigma_n^2} < \frac{3\alpha^2}{\pi^2}$, then the sequence $\{f_n\}_{n=1}^{\infty}$ satisfies the Restricted Isometry Property.*

Proof. From the definition of f_n, $\|f_n\| = 1$, for $n \geq 1$, then for any finite sequence of complex numbers $\{c_n\}_{n\geq 1}$ we have

$$\begin{aligned}
\Big\|\sum_{n=1}^{\infty} c_n f_n\Big\|^2 &= \sum_{m,n=1}^{\infty} c_n \bar{c}_m \langle f_n, f_m\rangle \\
&= \sum_{n=1}^{\infty} |c_n|^2 \big\|f_n\big\|^2 + \sum_{\substack{m,n=1 \\ m\neq n}}^{\infty} c_n \bar{c}_m \langle f_n, f_m\rangle \\
&\leq \sum_{n=1}^{\infty} |c_n|^2 + \sum_{\substack{m,n=1 \\ m\neq n}}^{\infty} |c_n \bar{c}_m \langle f_n, f_m\rangle| \\
&\leq \sum_{n=1}^{\infty} |c_n|^2 + \Bigg(\sum_{\substack{m,n=1 \\ m\neq n}}^{\infty} |c_n|^2|c_m|^2\Bigg)^{\frac{1}{2}} \Bigg(\sum_{\substack{m,n=1 \\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2\Bigg)^{\frac{1}{2}} \\
&\leq \sum_{n=1}^{\infty} |c_n|^2 + \Bigg(\sum_{n=1}^{\infty} |c_n|^2\Bigg)^{\frac{1}{2}} \Bigg(\sum_{m=1}^{\infty} |c_m|^2\Bigg)^{\frac{1}{2}} \Bigg(\sum_{\substack{m,n=1 \\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2\Bigg)^{\frac{1}{2}} \\
&= \sum_{n=1}^{\infty} |c_n|^2 + \sum_{n=1}^{\infty} |c_n|^2 \Bigg(\sum_{\substack{m,n=1 \\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2\Bigg)^{\frac{1}{2}} \\
&= \Bigg(1 + \Bigg(\sum_{\substack{m,n=1 \\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2\Bigg)^{\frac{1}{2}}\Bigg) \sum_{n=1}^{\infty} |c_n|^2 \\
&= (1+\delta) \sum_{n=1}^{\infty} |c_n|^2
\end{aligned}$$

where $\Bigg(\sum\limits_{\substack{m,n=1 \\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2\Bigg)^{\frac{1}{2}} = \delta$, by Lemma 4.1.

Similarly, we prove the first part of the inequality. We use the claim in equation (14) above, we have

$$\begin{aligned}
\Big\|\sum_{n=1}^{\infty} c_n f_n\Big\|^2 &\geq \Bigg(1 - \Bigg(\sum_{\substack{m,n=1 \\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2\Bigg)^{\frac{1}{2}}\Bigg) \sum_{n=1}^{\infty} |c_n|^2 \\
&= (1-\delta) \sum_{n=1}^{\infty} |c_n|^2.
\end{aligned}$$

Therefore, the sequence $\{f_n\}$ satisfies the Restricted Isometry Property for some $\delta \in (0,1)$, completing the proof. $\square$

If $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ is a given sequence, then for $\epsilon > 0$, we define a perturbation

sequence

$$\mathcal{M}_\epsilon := \left\{ \mu_n \in \mathbb{R} : \mu_n = \lambda_n + \epsilon_n,\, 0 < \epsilon_n \leq \epsilon\, \frac{k(\lambda_n, \lambda_n)}{\tau_n},\, n \geq 1 \right\}, \tag{15}$$

where $\tau_n = \max_{t \in [\lambda_n, \lambda_{n+1}]} k(t,t)$. In what follows, the constant A_f is the lower frame bound of the sequence $\{f_n\}$ in (9) and (11), and C_{Ber} is the Berntein constant from Lemma 2.2.

Theorem 4.3. *Given a de Branges space $\mathcal{H}(E)$, such that $E'/E \in \mathbb{H}^\infty(\mathbb{C}^+)$, and $\varphi(x)$ a phase function associated with E such that $\varphi'(x) \geq \alpha > 0$ on $\mathbb{R}$. If $\{f_n\}$ is a Riesz basis in $\mathcal{H}(E)$, then the sequence $\{\frac{k(\mu_n, z)}{\|k(\lambda_n, .)\|} : \mu_n \in \mathcal{M}_\epsilon\}$ is also a Riesz basis in $\mathcal{H}(E)$ whenever $\epsilon < \frac{\alpha A_f}{\pi C^2_{Ber}}$.*

Proof. Since the function $k(t,t)$ is continuous for all $t \in \mathbb{R}$, the Mean Value Theorem implies that there exists $t_n \in (\lambda_n, \mu_n)$ such that

$$\int_{\lambda_n}^{\mu_n} \frac{k(t,t)}{k(\lambda_n, \lambda_n)}\, dt = \epsilon_n \frac{k(t_n, t_n)}{k(\lambda_n, \lambda_n)}, \text{ for all } n \geq 1.$$

Moreover, since $\mu_n \in \mathcal{M}_\epsilon$, then

$$\epsilon_n \frac{k(t_n, t_n)}{k(\lambda_n, \lambda_n)} \leq \epsilon\, \frac{k(\lambda_n, \lambda_n)}{\tau_n} \frac{k(t_n, t_n)}{k(\lambda_n, \lambda_n)} \leq \epsilon, \text{for all } n \geq 1.$$

Let $f \in \mathcal{H}(E)$, and $h_n(z) := \frac{k(\mu_n, z)}{\|k(\lambda_n, .)\|}$, for $\mu_n \in \mathcal{M}_\epsilon$. Then

$$\begin{aligned}
|\langle f, f_n - h_n \rangle|^2 &= \frac{1}{k(\lambda_n, \lambda_n)}\, |f(\lambda_n) - f(\mu_n)|^2 \\
&= \frac{1}{k(\lambda_n, \lambda_n)} \left| \int_{\lambda_n}^{\mu_n} (f(t))'\, dt \right|^2 \\
&\leq \frac{1}{k(\lambda_n, \lambda_n)} \int_{\lambda_n}^{\mu_n} \left| \frac{f'(t)}{E(t)} \right|^2 \, dt \int_{\lambda_n}^{\mu_n} |E(t)|^2\, dt \\
&= \int_{\lambda_n}^{\mu_n} \left| \frac{f'(t)}{E(t)} \right|^2 \, dt \int_{\lambda_n}^{\mu_n} \pi\, \frac{k(t,t)}{k(\lambda_n, \lambda_n)} \frac{1}{\varphi'(t)}\, dt \\
&\leq \frac{\pi}{\alpha} \int_{\lambda_n}^{\mu_n} \left| \frac{f'(t)}{E(t)} \right|^2 \, dt \int_{\lambda_n}^{\mu_n} \frac{k(t,t)}{k(\lambda_n, \lambda_n)}\, dt \\
&\leq \frac{\pi\, \epsilon}{\alpha} \int_{\lambda_n}^{\mu_n} \left| \frac{f'(t)}{E(t)} \right|^2 \, dt.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} |\langle f, f_n - h_n \rangle|^2 &\leq \frac{\pi\, \epsilon}{\alpha} \int_{\mathbb{R}} \left| \frac{f'(t)}{E(t)} \right|^2 \, dt \\
&= \frac{\pi\, \epsilon}{\alpha} \|f'/E\|^2 \\
&\leq \frac{\pi\, \epsilon}{\alpha} C^2_{\text{Ber}}\, \|f\|^2,
\end{aligned}$$

where the last inequality follows from Lemma 2.2. Consequently, $\{h_n\}$ is a Riesz basis by Lemma 3.1 with $R = \frac{\pi\epsilon}{\alpha} C^2_{\text{Ber}} < A_f$ by the hypothesis.

□

Theorem 4.4. *Let $\mathcal{H}(E)$ be a de Branges space, with reproducing kernel function $k(w,z)$. Let $\{\lambda_n\}, \{\mu_n\}$ be two sequences of real numbers, and $\{h_n(z) := \frac{k(\mu_n,z)}{\|k(\lambda_n,.)\|}\}$ be a Riesz basis in $\mathcal{H}(E)$ with frame bounds A_h and B_h. If there exits positive constants C_1, C_2 such that*

$$C_1 k(\lambda_n, \lambda_n) \le k(\mu_n, \mu_n) \le C_2\, k(\lambda_n, \lambda_n), \tag{16}$$

for all $n \ge 1$, then the sequence $\{\frac{k(\mu_n,z)}{\|k(\mu_n,.)\|}\}$ is also a Riesz basis in $\mathcal{H}(E)$, whenever $CB_h < A_h$, where $C = (1 + \frac{1}{C_1} - \frac{2}{\sqrt{C_2}})$.

Proof. Since the sequence $\{h_n\}$ is a Riesz basis, then for all $f \in \mathcal{H}(E)$,

$$A_h \|f\|^2 \le \sum_{n=1}^{\infty} |\langle f, h_n\rangle|^2 \le B_h \|f\|^2.$$

Let $f \in \mathcal{H}(E)$, and $g_n(z) := \frac{k(\mu_n,z)}{\|k(\mu_n,.)\|}$. Then

$$\begin{aligned}
|\langle f, h_n - g_n\rangle|^2 &= \left| \frac{f(\mu_n)}{\sqrt{k(\lambda_n,\lambda_n)}} - \frac{f(\mu_n)}{\sqrt{k(\mu_n,\mu_n)}} \right|^2 \\
&= |f(\mu_n)|^2 \left| \frac{1}{\sqrt{k(\lambda_n,\lambda_n)}} - \frac{1}{\sqrt{k(\mu_n,\mu_n)}} \right|^2 \\
&= |f(\mu_n)|^2 \left| \frac{1}{k(\lambda_n,\lambda_n)} + \frac{1}{k(\mu_n,\mu_n)} - \frac{2}{\sqrt{k(\lambda_n,\lambda_n)k(\mu_n,\mu_n)}} \right| \\
&\le R \frac{|f(\mu_n)|^2}{k(\lambda_n,\lambda_n)}
\end{aligned}$$

where $R = 1 + \frac{1}{C_1} - \frac{2}{\sqrt{C_2}}$. Thus, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} |\langle f, h_n - g_n\rangle|^2 &\le R \sum_{n=1}^{\infty} \frac{|f(\mu_n)|^2}{k(\lambda_n,\lambda_n)} \\
&= R \sum_{n=1}^{\infty} |\langle f, h_n\rangle|^2 \\
&\le R B_h \|f\|^2.
\end{aligned}$$

Consequently, $\{g_n\}$ is a Riesz basis by Lemma 3.1 as $RB_h < A_h$. □

Now we state the main result on stability of Riesz basis in de Branges spaces, the proof is an immediate consequence of Theorem 4.3 and Theorem 4.4.

Theorem 4.5. *Given a de Branges space $\mathcal{H}(E)$, such that $E'/E \in \mathbb{H}^\infty(\mathbb{C}^+)$, and $\varphi(x)$ a phase function associated with E such that $\varphi'(x) \geq \alpha > 0$ on $\mathbb{R}$. Let $\{f_n\}$ be a Riesz basis in $\mathcal{H}(E)$ with bounds A_f, B_f. Let $\mathcal{M}_\epsilon$ be the sequence defined in (15), and assume that there exits positive constants C_1, C_2 such that*

$$C_1 k(\lambda_n, \lambda_n) \leq k(\mu_n, \mu_n) \leq C_2\, k(\lambda_n, \lambda_n), \text{ for all } n \geq 1. \tag{17}$$

Then the sequence $\{\frac{k(\mu_n,z)}{\|k(\mu_n,.)\|} : \mu_n \in \mathcal{M}_\epsilon\}$ is also a Riesz basis in $\mathcal{H}(E)$ whenever

$$\epsilon < \frac{\alpha A_f}{\pi C^2_{_{Ber}}} \text{ and } C\, B_f(1+\sqrt{R/B_f}\ )^2 < A_f(1-\sqrt{R/A_f}\ )^2$$

where $R = \frac{\pi\,\epsilon}{\alpha} C^2_{_{Ber}}$ and $C = (1+\frac{1}{C_1} - \frac{2}{\sqrt{C_2}})$.

Remark 4.1. *de Branges spaces $\mathcal{H}(E)$ that satisfy the conditions of the previous theorems in general do not have simple analytic characterizations. We would like to emphasize that the best way to construct the corresponding generating functions $E \in \mathcal{HB}$ is via their Weierstrass factorization formula. A special class of Hermite-Biehler functions is the Pólya class where any function can be characterized by its Hadamard factorization formula. For the sake of completeness, we include some examples of such functions, see [1] and [13]:*

(1) *Let E have the form*

$$E(z) = \gamma\, e^{bz} e^{-iaz} \prod_{n\in\mathbb{Z}} \left(1 - \frac{z}{z_n}\right) e^{zRe(\frac{1}{z_n})}, \tag{18}$$

and let the zeros z_n satisfy the following conditions:

(a). *$z_n = \beta n + w_n$, for all $n \in \mathbb{Z}$, where $\beta > 0$, and the sequence $\{w_n\}_{n\in\mathbb{Z}}$ is bounded,*

(b). *$Im(w_n) \geq \alpha > 0$.*

Then $\frac{E'}{E} \in \mathbb{H}^\infty(\mathbb{C}^+)$. If, in addition, $w_n = u_n + iv_n$ where $u_n \in [\alpha_1, \alpha_2]$ and $v_n \in [a_1, a_2]$, $a_1 > 0$ for all $n \in \mathbb{Z}$, then $E'/E \in \mathbb{H}^\infty(\mathbb{C}^+)$. and $\varphi'(x)$ is bounded away from zero.

(2) *Let*

$$E(z) = \gamma e^{-iaz} S(z) \prod_{n=1}^{\infty} \left(1 - \frac{z}{\bar{z}_n}\right) e^{h_n z},$$

for all $z \in \mathbb{C}$, where the sequence $\{z_n\}_{n=1}^\infty \subset \mathbb{C}^+$ has no condensation points in $\mathbb{C}$ and satisfies the Blaschke condition

$$\sum_{n=1}^{\infty} y_n / \left(x_n^2 + y_n^2\right) < +\infty,$$

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which guarantee the convergence of the previous product, and

$$h_n = x_n / \left(x_n^2 + y_n^2 \right), \; n \in \mathbb{N},$$

$a > 0$, S is an entire function taking the real values on the real line and having only real zeros, and γ is a complex number with modulus 1. If the sequence $\{z_n\}_{n=1}^{\infty}$ is contained in the set $\Gamma_\tau = \{z \in \mathbb{C}^+ : \tau < \arg z < \pi - \tau\}$, $\tau > 0$, then $\frac{E'}{E} \in \mathbb{H}^\infty(\mathbb{C}^+)$ and $\varphi'(x)$ is bounded away from zero.

Furthermore, a wide class of de Branges spaces for which the previous theorems may be applied is the homogeneous *de Branges spaces. Such spaces are related to the classical Bessel functions and more general confluent hypergeometric functions, and were characterized by L. de Branges [12, 13]. We present a brief review of the construction of these spaces. Let $\nu > -1$. A space $\mathcal{H}(E)$ is said to be* homogeneous *of order ν if, for all $0 < a < 1$ and all $F \in \mathcal{H}(E)$, the function $z \mapsto a^{\nu+1}F(az)$ belongs to $\mathcal{H}(E)$ and has the same norm as F. For $\nu > -1$ consider the real entire functions $A_\nu(z) : \mathbb{C} \to \mathbb{C}$ and $B_\nu(z) : \mathbb{C} \to \mathbb{C}$ given by*

$$A_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}z\right)^{2n}}{n!(\nu+1)(\nu+2)\dots(\nu+n)} = \Gamma(\nu+1)\left(\frac{1}{2}z\right)^{-\nu} J_\nu(z)$$

and

$$B_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}z\right)^{2n+1}}{n!(\nu+1)(\nu+2)\dots(\nu+n+1)} = \Gamma(\nu+1)\left(\frac{1}{2}z\right)^{-\nu+1} J_\nu(z)$$

where

$$J_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n \left(\frac{1}{2}z\right)^{2n+\nu}}{n!\Gamma(\nu+n+1)}$$

is the classical Bessel function of the first kind. These special functions have only real, simple zeros and have no common zeros. Furthermore, they satisfy the following differential equations

$$A'_\nu(z) = -B_\nu(z) \quad \textit{and} \quad B'_\nu(z) = A_\nu(z) - (2\nu+1)B_\nu(z)/z. \tag{19}$$

If we define

$$E_\nu(z) := A_\nu(z) - iB_\nu(z),$$

then the function $E_\nu(z)$ is a Hermite-Biehler function with no real zeros, of bounded type in the upper-half, and is of exponential type 1 in $\mathbb{C}$. Also we have that

$$c_\nu |x|^{2\nu+1} \leq |E_\nu(x)|^{-2} \leq C_\nu |x|^{2\nu+1},$$

for all real $|x| \geq 1$ and for some $c_\nu, C_\nu > 0$, see [15]. Moreover, it is known that $A_\nu, B_\nu \notin \mathcal{H}(E_\nu)$. Note that if $\nu = -1/2$ we have $A_{-1/2}(z) = \cos z$ and

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$B_{-1/2}(z) = \sin z$, hence, $E_{-1/2}(z) = e^{-iz}$ and the space $\mathcal{H}(E_{-1/2})$ coincides with the Paley-Wiener space $\mathcal{P}W_1$. By (19) we have

$$i\frac{E_\nu'(z)}{E_\nu(z)} = 1 - (2\nu+1)\frac{B_\nu(z)}{zE_\nu(z)},$$

for all $z \in \mathbb{C}^+$. Hence $E_\nu'(z)/E_\nu(z) \in H^\infty(\mathbb{C}^+)$. This also implies that the phase function $\varphi_\nu(z)$ associated with $E_\nu(z)$ satisfies

$$\varphi_\nu'(x) = 1 - \frac{(2\nu+1)A_\nu(x)B_\nu(x)}{x\,|E_\nu(x)|^2}.$$

Hence, $\varphi_\nu'(x) \simeq 1$ for all real x.

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Conflict of interest

The authors declare that they have no conflict of interest.

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Solving the linear moment problems for nonhomogeneous linear recursive sequences

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Abstract

The present paper aimed to explore the linear moment problem for the real sequences defined by the nonhomogeneous linear recursive relation. Various properties are provided, especially, those related to the Hankel matrices. Some considerations in connection with K-moment problem, for the nonhomogeneous recursive are discussed.

Keywords: Linear moment problem, $K-$moment problem, Hankel matrix, nonhomogeneous linear recursive sequences.

1 Introduction

In view of its fundamental role in various fields of mathematics and applied science, the linear moment problem has been extensively studied in the literature (see [4,5,9,11–13]). Especially, it has been shown that this problem is useful for some topics in physics, such that the quantum dynamical systems, the resolvent $R_\varphi(\lambda)$ of a given Hamiltonian A, which can be written as an infinite series in terms of $1/\lambda$, whose coefficients are the moment $\mu_n = \langle\varphi|A^n|\varphi\rangle$ of order n of the operator A, where φ is a state vector of the given system (see [4,12] for example). Furthermore, the linear moment problem is also related to the Lanczos numerical method, which is an important technique for finding the positions of n particles such that the first $2n-1$ moments own given values (see [5, 13] for example).

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Recently, the linear moment problem has been investigated in the literature, by various methods (see, for example, [4, 9, 11, 12]).

The linear moment problem is simple to formulate. Indeed, let $\mathcal{H}$ be a real separable Hilbert space, $\mathcal{L}(\mathcal{H})$ be the space of linear operators on $\mathcal{H}$ and $\mathcal{S}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ the subspace of self adjoint operators on $\mathcal{H}$. For a given operator $A \in \mathcal{L}(\mathcal{H})$ and non-vanishing $x \in \mathcal{H}$, the sequence $\Gamma = \{\alpha_n\}_{n\geq 0}$ defined by $\alpha_n = \langle A^n x|x\rangle$ for $n \geq 0$, is called the moment sequence of A on x, and α_n is the moment of order n of the operator A on x. The linear moment problem is the reciprocal of the previous situation. More precisely, let $\Gamma = \{\alpha_n\}_{0\leq n\leq p}$ ($p \leq +\infty$) be a sequence of real numbers, the linear moment problem associated with Γ consists to find a self-adjoint operator $A \in \mathcal{S}(\mathcal{H})$ and a non-vanishing vector $x \in \mathcal{H}$ such that,

$$\alpha_n = \langle A^n x|x\rangle, \quad \text{for } 0 \leq n \leq p. \tag{1}$$

The problem (1) is called the *full linear moment problem* when $p = +\infty$ and the *truncated linear moment problem* for $p < +\infty$ (see [7–9, 12], for example).

On the other hand, the linear moment problem (1) for the sequence Γ, is also related to the classical power K-moment problem (K is a closed set of $\mathbb{R}$), whose aim is to find a positive Borelean measure μ with $\text{supp}(\mu) \subset K$ such that

$$\alpha_n = \int_K t^n d\mu(t), \quad \text{for } 0 \leq n \leq p, \tag{2}$$

where $p \leq +\infty$. The moment problem (2) is important in operator theory, particularly, it is related to the study of the shift of subnormal operators and subnormal extension (see [1, 3, 6–8]). Recently, the two preceding moment problems (1) and (2) have been studied in [3, 9–11], for some sequences defined by linear recursive relations. Moreover, it was established the closed connection between the full and the truncated moment problem for recursive sequences in [9, 11]. More precisely, let $\{u_n\}_{n\geq 0}$ be the sequence satisfying the following linear recursive relation of order r,

$$u_{n+1} = a_0 u_n + a_1 u_{n-1} + \cdots + a_{r-1} u_{n-r+1} \text{ for } n \geq r-1, \tag{3}$$

where $u_0, u_1, \ldots, u_{r-1}$ are the initial data, it was shown in [9–11] that, for the linear moment problems (1), the full one ($p = +\infty$) and the truncated one ($p < +\infty$) are closely related. Especially, it was shown in [9] that in the finite dimensional case ($\dim_{\mathbb{R}} \mathcal{H} < +\infty$), the two preceding linear moment problems (the full and the truncated) are identical. On the other side, it was shown in [9] that the full and truncated moment problem (2), for the recursive sequence (3), are equivalent.

The purpose of this paper is to study the linear moment problem (1), for a real non-homogeneous recursive sequence $\{v_n\}_{n\geq 0}$ of order r, defined by the following recursive relation,

$$v_{n+1} = a_0 v_n + a_1 v_{n-1} + \cdots + a_{r-1} v_{n-r+1} + c_{n+1} \text{ for } n \geq r-1, \tag{4}$$

where the coefficients $a_0, \ldots, a_{r-1}$ ($r \geq 2$, $a_{r-1} \neq 0$) are real numbers, $v_0 = \alpha_0, \ldots, v_{r-1} = \alpha_{r-1}$ are the initial values, and $\mathcal{C} = \{c_n\}_{n\geq r}$ is a (non trivial) real sequence. It seems to us that properties of the linear moment problem (1) for nonhomogeneous sequences (4), can be useful for the study of certain related perturbed physical systems. For the K-moment problem (2), it can be also, for studying the perturbed moment, of the shift of operators.

In this study, we characterize the solution of the linear moment problem (1) for sequences (4) in the general setting, especially, when the operator $A \in \mathcal{S}(\mathcal{H})$, namely, A is self-adjoint. When the real separable Hilbert space $\mathcal{H}$ is of finite dimension and the non-homogeneous sequence $\{v_n\}_{n\geq 0}$ is a moment sequence of an operator A, on a non-vanishing $x \in \mathcal{H}$, we establish that the sequence $\{c_n\}_{n\geq r}$ is a linear recursive sequence of type (3). And when the real separable Hilbert space $\mathcal{H}$ is of infinite dimension and the non-homogeneous sequence $\{v_n\}_{n\geq 0}$ is a moment sequence of an operator A, on a non-vanishing $x \in \mathcal{H}$, then the general term of the sequence $\{c_n\}_{n\geq r}$, is expressed as a limit of $c_n = \lim_{s\to+\infty} c_n^{(s)}$, where $c_n^{(s)}$ is a linear recursive sequence of type (3). We establish the solution of the linear moment problem (1), using the properties of the Hankel matrices. The special case when $\{c_n\}_{n\geq r}$ is a linear recursive sequence of type (3), is discussed. Moreover, the K-moment problem (2) for nonhomogeneous recursive sequences (4) is provided, using the spectral measures of self-adjoint operators. By the way, some other consequences are derived, especially, the Stieltjes and Hamburger moment problems (2), for the nonhomogeneous recursive sequences (4), are discussed through the spectral measures of self-adjoint operators. It should be noted that the study of these two problems for the sequences (4), is not common in the literature.

2 Linear moment problem and sequences (4)

Let improve the connections between solutions of (4) considered as a difference equation and the linear moment problem (1). Let $\{Q_n\}_{n\geq r}$ be the family of polynomials defined by $Q_n(z) = z^{n-r}P(z)$, where $P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$, is the so-called characteristic polynomial of the homogeneous part of the sequence (4). Let $x \neq 0$ be an element of $\mathcal{H}$ and $A \in \mathcal{S}(\mathcal{H})$. Suppose that $v_n = \langle A^n x|x\rangle$, for every $n \geq 0$. Then, we have, $\langle A^{n+1}x|x\rangle = a_0\langle A^n x|x\rangle + \cdots + a_{r-1}\langle A^{n-r+1}x|x\rangle + c_{n+1}$, for every $n \geq r-1$. Therefore, we derive $c_{n+1} = \langle Q_{n+1}(A)x|x\rangle$, for every $n \geq r-1$. Consequently, we can state the following proposition.

Proposition 2.1. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence (4), of characteristic polynomial $P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$. Suppose that $\mathcal{T} = \{v_n\}_{n\geq 0}$, is a moment sequence of an operator $A \in \mathcal{S}(\mathcal{H})$, namely, $v_n = \langle A^n x|x\rangle$, for every $n \geq 0$, where $x \neq 0$. Then, the sequence $\{c_n\}_{n\geq r}$ is given by $c_{n+1} = \langle Q_{n+1}(A)x|x\rangle$, for every $n \geq r-1$, where $Q_n(z) = z^{n-r}P(z)$.*

Therefore, the question of studying the converse of the preceding affirmation of Proposition 2.1 arises.

Theorem 2.2. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence (4), of characteristic polynomial $P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$. Let $A \in \mathcal{S}(\mathcal{H})$ and $x \neq 0 \in \mathcal{H}$. Then, we have $v_n = \langle A^n x|x\rangle$, for every $n \geq 0$, if and only if, $v_n = \langle A^n x|x\rangle$ for $n = 0, 1, \ldots, r-1$ and $c_n = \langle A^{n-r}P(A)x|x\rangle$, for $n \geq r$.*

Proof. Suppose $v_n = \langle A^n x|x\rangle$ $(n \geq 0)$, for some $x \neq 0$ in $\mathcal{H}$ and $A \in \mathcal{S}(\mathcal{H})$. Then, we have $c_k = v_k - \sum_{j=0}^{r-1} a_j v_{k-j-1} = \left\langle (A^k - \sum_{j=0}^{r-1} a_j A^{k-j-1})x|x \right\rangle = \langle A^{k-r}P(A)x|x\rangle$, for every $k \geq r$. Conversely, suppose that $v_n = \langle A^n x|x\rangle$, for $n = 0, 1, \ldots, r-1$ and $c_n = \langle A^{n-r}P(A)x|x\rangle$ for every $n \geq r$. Therefore, we have

$$v_r = \sum_{j=0}^{r-1} a_j \langle A^{r-j-1}x|x\rangle + \langle P(A)x|x\rangle = \langle A^r x|x\rangle.$$

And, by induction, we derive that $v_n = \langle A^n x|x\rangle$, for every $n \geq 0$. □

As a consequence of Theorem 2.2, we obtain the following corollary.

Corollary 2.3. *Let $A \in \mathcal{S}(\mathcal{H})$ and $x \in \mathcal{H}$, then under the data of Theorem 2.2, the following statements are equivalent,*

(i) $v_n = \langle A^n x|x\rangle$, *for every* $n \geq 0$.

(ii) $v_n = \langle A^n x|x\rangle$, *for* $n = 0, 1, \ldots, 2r-1$, *and* $c_n = \sum_{j=0}^{r-1} a_j c_{n-j-1} + \langle A^{n-2r}z|z\rangle$ *for every* $n \geq 2r$, *where* $z = P(A)x$.

Proof. It suffices to establish the equivalence between (ii) and the second statement of Theorem 2.2. Let A be a self-adjoint operator, suppose that $v_n = \langle A^n x|x\rangle$ for $n = 0, 1, \ldots, r-1$ and $c_n = \langle A^{n-r}P(A)x|x\rangle$, for every $n \geq r$. Then, for $z = P(A)x$, we have, $\langle A^{n-2r}z|z\rangle = \langle A^{n-r}x|P(A)x\rangle - \sum_{j=0}^{r-1} a_j \langle A^{n-r-j-1}P(A)x|x\rangle = c_n - \sum_{j=0}^{r-1} a_j c_{n-j-1}$, for any $n \geq 2r$. Conversely, suppose that (ii) holds. A direct computation shows that $c_n = \langle A^{n-r}P(A)x|x\rangle$, for $n = r, r+1, \ldots, 2r-1$. On the other hand, by induction we prove that $c_n = \langle A^{n-r}P(A)x|x\rangle$, for every $n \geq 2r$. It follows that (i) and (ii) are equivalent. □

We conclude this section by the following observation. Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence (4), whose characteristic polynomial is $P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$. Suppose that there exist $A \in \mathcal{S}(\mathcal{H})$ and $x \in \mathcal{H}$ such that $v_n = \langle A^n x|x\rangle$. Then, we have, $c_{2k} - \sum_{j=0}^{r-1} a_j c_{2k-j-1} = \|A^{k-r}P(A)x\|^2$ for every $k \geq r$. Therefore, when $c_{2k} \neq 0$, for some $k \in \mathbb{N}$, we have $c_{2k} > \sum_{j=0}^{r-1} a_j c_{2k-j-1}$, for any $k \geq r$. This later inequality is a necessary condition for the existence of the solution of the linear moment problem (1), for the sequence $\mathcal{T} = \{v_n\}_{n\geq 0}$ defined by (4).

3 The linear moment problem (1) for sequences (4)

Let $\mathcal{H}$ be a finite dimensional Hilbert space over $\mathbb{R}$ ($m = \dim_{\mathbb{R}} \mathcal{H}$) and $\mathcal{T} = \{v_n\}_{n\geq 0}$ a sequence (4). A straightforward computation and by using Theorem 2.2, allows us to see that $\mathcal{T} = \{v_n\}_{n\geq 0}$ is a moment sequences of a self-adjoint operator A on a non-vanishing vector x of $\mathcal{H}$ if and only if $v_n = \sum\limits_{j=1}^{s} {\lambda_j}^n \|x_j\|^2$ for $n = 0, 1, \ldots, r-1$ and

$$c_n = \sum_{j=1}^{s} \frac{P(\lambda_j)}{\lambda_j^r} \|x_j\|^2 {\lambda_j}^n, \tag{5}$$

where $x_j = \Pi_j x \in \mathcal{H}_j$ $(0 \leq j \leq s)$, the subspace of the eigenvectors of A, corresponding to the eigenvalues λ_j $(0 \leq j \leq s)$. Expression (5) is nothing else but the analytic formula of the sequence $\{c_n\}_{n\geq r}$, viewed as a linear recursive sequence of type (3) of order s. More precisely, (5) implies that $\{c_n\}_{n\geq r}$ is a linear recursive sequence of type (3), of characteristic polynomial $K(z) = \prod\limits_{j=1}^{s} (z - \lambda_j)$. Thus, we can state the following proposition.

Proposition 3.1. *Let $\mathcal{T}$ be a sequence* (4). *Suppose that $\mathcal{T}$ is a moment sequences of a self-adjoint operator A on the finite dimensional Hilbert space $\mathcal{H}$. Then, the nonhomogeneous part $\mathcal{C}$ is a linear recursive sequence of type* (3) *of order s (with $s \leq \dim \mathcal{H}$). More precisely, the characteristic polynomial of $\mathcal{C}$ is $K(z) = \prod\limits_{j=1}^{s} (z - \lambda_j)$, where the λ_j $(0 \leq j \leq s)$ are the eigenvalues of A.*

Suppose that $\mathcal{H}$ is a separable real Hilbert space (over $\mathbb{C}$) of infinite dimension. The simplest spectral theorem (after the algebraic case) concerns a compact self-adjoint and a compact normal operator A on $\mathcal{H}$, and asserts that $\mathcal{H}$ coincide with the closure of the orthogonal sum of the eigenspaces $\mathcal{H}_n$, corresponding to all possible eigenvalues $\{\lambda_n\}_{n\geq 0}$. With a view to generalization it is convenient to express it under the spectral resolution form $Ax = \sum\limits_{n=0}^{+\infty} \lambda_n \Pi_n x$, where Π_n is an orthoprojection onto $\mathcal{H}_n$, the eigenspace corresponding to the eigenvalue λ_j, and $x = \sum\limits_{n=0}^{+\infty} \Pi_n x$. We consider the class of operators satisfying the Spectral Theorem, which are called spectral operators or S-operators for short.

Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence (4), with characteristic polynomial P. Suppose that $\mathcal{T}$ is a sequence of moments of an S-operator A of $\mathcal{L}(\mathcal{H})$, on a non-vanishing vector $x \in \mathcal{H}$, namely, $v_n = \langle A^n x | x \rangle$, for every $n \geq 0$, where A is an S-operator and $x = \sum\limits_{n=0}^{+\infty} \Pi_n x \in \mathcal{H}$.

Let $s \geq 1$ and consider the sequence $\{v_n^{(s)}\}_{n\geq 0}$ defined as follows: $v_j^{(s)} = v_j$

for $i = 0, 1, \ldots, r-1$, and

$$v_{n+1}^{(s)} = a_0 v_n^{(s)} + a_1 v_{n-1}^{(s)} + \cdots + a_{r-1} v_{n-r+1}^{(s)} + c_{n+1}^{(s)}, \tag{6}$$

for $n \geq r-1$, where $c_n^{(s)} = \sum_{p=0}^{s} \frac{P(\lambda_p)}{\lambda_p^r} \|x_p\|^2 {\lambda_p}^n$. It is easy to see that $c_n = \lim_{s\to+\infty} c_n^{(s)}$. For $n = r$, expression (6) shows that we have $v_r = \lim_{s\to+\infty} v_r^{(s)}$. By induction on n, we have $v_n = \lim_{s\to+\infty} v_n^{(s)}$, for every $n \geq r$. In conclusion, we have the following result.

Theorem 3.2. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4)*, with characteristic polynomial P. Suppose the Hilbert space $\mathcal{H}$ is of infinite dimension and that $\mathcal{T}$ is a moment sequences of an S-operator A on $\mathcal{H}$, on a non-vanishing vector $x = \sum_{n=0}^{+\infty} \Pi_n x$. Then, we have $v_n = \lim_{s\to+\infty} v_n^{(s)}$, for every $n \geq r$, where $\{v_n^{(s)}\}_{n\geq 0}$ is a sequence* (4)*, whose associate nonhomogeneous term is*

$$c_n^{(s)} = \sum_{p=0}^{s} \frac{P(\lambda_p)}{\lambda_p^r} \|x_p\|^2 {\lambda_p}^n, \tag{7}$$

where $P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$ ($a_{r-1} \neq 0$) is the characteristic polynomial of $\mathcal{T}$ and $x_p = \Pi_p x \in \mathcal{H}$. Moreover, expression (7) *stands for the analytic formula of the sequence $\{c_n^{(s)}\}_{n\geq 0}$, viewed as a linear recursive sequence of type* (3)*.*

From Theorem 3.2, we derive that

$$c_n = \sum_{p=0}^{+\infty} \frac{P(\lambda_p)}{\lambda_p^r} \|x_p\|^2 {\lambda_p}^n. \tag{8}$$

Remark 3.3. If there exists $s \geq 1$ such that $\lambda_p = 0$, for every $p \geq s+1$, we show that expressions (5) and (8) are identical. Suppose that for every $N > 0$ there exists $k \geq N$ such that $\lambda_k \neq 0$. Therefore, expression (8) doesn't represent a recursive sequence of finite order. Meanwhile, we can approximate this situation by a family of sequences (4), whose associated c_n is given by expression (7).

4 Hankel matrices and solution of the linear moment problem (1)

In this section, we present algebraic treatment of the Hankel matrix related to the sequences defined by (4), and its use for characterizing the existence of solutions for the linear moment problem (1).

Let H_k be the Hankel matrix of size $k+1$, whose entries are defined from the elements of the sequence $\mathcal{T} = \{v_i\}_{i\geq 0}$, in the sense that $H_k := (v_{i+j})_{0\leq i,j\leq k}$.

The j^{th} column of H_k will be denoted by $\mathbf{V}_j := (v_{j+\ell})_{\ell=0}^{k}, 0 \leq j \leq k$, so that H_k can be briefly written as $H_k = (\mathbf{V}_0 \;\; \mathbf{V}_1 \;\; \cdots \;\; \mathbf{V}_k)$. Observe that we can verify that

$$\mathbf{V}_{r+k} = a_0\mathbf{V}_{r+k-1} + a_1\mathbf{V}_{r+k-2} + \cdots + a_{r-1}\mathbf{V}_k + \widehat{\mathbf{C}}_{r+k}, \tag{9}$$

where $\widehat{\mathbf{C}}_{r+k} := (c_{r+\ell})_{\ell=0}^{r+k-1}$.
With a vectorial representation, we can write the matrix H_{r+n} as follows

$$H_{r+n} = \left(\mathbf{V}_0 \quad \mathbf{V}_1 \quad \cdots \quad \mathbf{V}_{r-1} \quad \middle| \quad \mathbf{V}_r \quad \cdots \quad \mathbf{V}_{r+k} \quad \cdots \quad \mathbf{V}_{r+n-1}\right).$$

Using expression (9) and some computational techniques emanated from determinant properties, we get,

$$\det H_{r+n} = \det\left(\mathbf{V}_0 \quad \mathbf{V}_1 \quad \cdots \quad \mathbf{V}_{r-1} \quad \middle| \quad \widehat{\mathbf{C}}_r \quad \cdots \quad \widehat{\mathbf{C}}_{r+k} \quad \cdots \quad \widehat{\mathbf{C}}_{r+n-1}\right).$$

Repeating the same treatment on the matrix $S_k := (v_{i+j+1})_{0\leq i,j\leq k}$, one gets out of it by the following result.

Proposition 4.1. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4)*,*

$$H_{r+n} = (v_{i+j})_{0\leq i,j\leq r+n-1} \text{ and } S_{r+n} = (v_{i+j+1})_{0\leq i,j\leq r+n-1}$$

be the Hankel matrices associated with $\mathcal{T}$. Then, we have

$$\det H_{r+n} = \begin{vmatrix} v_0 & \cdots & v_{r-1} & c_r & \cdots & c_{r+n-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{r-1} & \cdots & v_{2r-2} & c_{2r-1} & \cdots & c_{2r+n-2} \\ v_r & \cdots & v_{2r-1} & c_{2r} & \cdots & c_{2r+n-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{r+n-1} & \cdots & v_{2r+n-2} & c_{2r+n-1} & \cdots & c_{2r+2n-2} \end{vmatrix} \tag{10}$$

and

$$\det S_{r+n} = \begin{vmatrix} v_1 & \cdots & v_r & c_{r+1} & \cdots & c_{r+n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_r & \cdots & v_{2r-1} & c_{2r} & \cdots & c_{2r+n-1} \\ v_{r+1} & \cdots & v_{2r} & c_{2r+1} & \cdots & c_{2r+n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{r+n} & \cdots & v_{2r+n-1} & c_{2r+n} & \cdots & c_{2r+2n-1} \end{vmatrix}. \tag{11}$$

Expression (10) shows that, for $n \geq 0$, it appears only the columns which depend on the entries of the sequence $\{c_n\}_{n\geq r}$ after the r-th column, in the determinant of the Hankel matrix H_{r+n}. A similar situation is observed for the matrix $S_k = (v_{i+j+1})_{0\leq i,j\leq k}$.

If the sequence $\mathcal{C} = \{c_n\}_{n\geq r}$ is also of type (3) of order s, then the $r+s-th$ column of the matrix H_{r+n} is a linear combination of the columns $r, r+1, \ldots, r+s-1$, and the $r+s+1-th$ column of the matrix S_{r+n} is a linear combination of the columns $r+1, r+2, \ldots, r+s$. Therefore, by Proposition 4.1, we get the following property.

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Proposition 4.2. *If the sequence $\{c_n\}_{n\geq r}$ is also a linear recursive sequence of type* (3) *of order s, then we have,*

1. $\det H_{r+n} = 0$*, for $n \geq s$, if and only if, the $r+s+1$-column of the matrix H_{r+n} is a linear combination of the previous s columns, namely, the $r, r+1, \ldots, r+s-1$ columns of the matrix H_{r+n}.*

2. $\det S_{r+n} = 0$*, for $n \geq s+1$, if and only if, the $s+1$-column of the matrix S_{r+n} is a linear combination of the previous s columns, namely, the $r+1, r+2, \ldots, r+s$ columns of the matrix H_{r+n}.*

The two Hankel matrices $H_{r+n} = (v_{i+j})_{0\leq i,j\leq r+n-1}$ and $S_{r+n} = (v_{i+j+1})_{0\leq i,j\leq r+n-1}$ and their determinants (10)-(11), play a central role for solving the two moment problems (1)-(2) and their applications.
We recall that it was established in [12, Lemma 1.1] that a $N \times N$ Hermitean matrix A is strictly positive definite if and only if each sub-matrix $A_k = (a_{ij})_{1\leq i,j\leq k}$ has $\det(A_k) > 0$, for $k = 1, 2, \ldots, N$. For a given Hankel matrix $H = (m_{i+j})_{i,j\geq 0}$, we consider the family of sub-matrices $H_n = (m_{i+j})_{0\leq i,j\leq n}$. Then, [12, Proposition 1.2] shows that for a Hankel matrix the family of sesquilinear form $\mathcal{F} = \{\mathtt{H}_n\}_{n\geq 0}$, defined by $\mathtt{H}_n(\alpha,\beta) = \sum_{j,k=0}^{n} m_{j+k}\alpha_j\bar{\beta}_k$, is (strictly) positive definite if and only if $\det(H_n) > 0$, where $H_n = (m_{i+j})_{0\leq i,j\leq n}$. Equivalently, we say that the Hankel matrix $H_n = (m_{i+j})_{0\leq i,j\leq n}$ is positive definite if and only if $\det(H_n) > 0$, where $H_n = (m_{i+j})_{0\leq i,j\leq n}$.

In order to establish the existence of solution of the linear moment problem (1), we will present a result of the closed relation between Hankel positive matrix, self-adjoint operator and measure. More precisely, we recall that from [6] the following theorem.

Theorem 4.3. *If $\{v_n\}_{n\geq 0}$ is a sequence of real numbers, the following statements are equivalent.*

(a) There is a self-adjoint operator A and a vector e such that $e \in \operatorname{dom} A^n$ for all n and $v_n = \langle A^n e, e\rangle$, for all $n \geq 0$.

(b) If $\alpha = (\alpha_0, \ldots, \alpha_n)$, where $\alpha_j \in \mathbb{C}$, then we have $\displaystyle\sum_{j,k=0}^{n} m_{j+k}\alpha_j\bar{\alpha}_k \geq 0$, for every $n \geq 0$.

(c) There is a positive regular Borelean measure μ on $\mathbb{R}$ such that $\displaystyle\int |t|^n d\mu(t) < \infty$ for all $n \geq 0$ and $v_n = \displaystyle\int t^n d\mu(t)$.

Therefore, for the Hankel matrix $H = (m_{i+j})_{i,j\geq 0}$, the second assertion of Theorem 4.3, implies that the sesquilinear form defined by $\mathtt{H}_n(\alpha,\beta) = \sum_{j,k=0}^{n} m_{j+k}\alpha_j\bar{\beta}_k$, is a (strictly) positive definite form if and only if the matrix $H_n = (m_{i+j})_{0\leq i,j\leq n}$ is (strictly) positive definite, for every $n \geq 0$. Equivalently, the second assertion

of Theorem 4.3, shows that the Hankel matrix $H = (m_{i+j})_{i,j\geq 0}$ is positive, or in an equivalent way, $\det H_n \geq 0$, for every $n \geq 0$, where $H_n = (m_{i+j})_{0\leq i,j\leq n}$.

Combining Proposition 4.1 and Theorem 4.3, we can formulate the following result.

Theorem 4.4. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4). *Then, the following assertions are equivalent,*

1. *The linear moment problem* (1) *for sequence* (4) *owns a solution.*
2. *The Hankel matrix $H = (v_{i+j})_{i,j\geq 0}$ is positive.*
3. $\det H_n \geq 0$*, for every $0 \leq n \leq r-1$ and* $\det H_{n+r} \geq 0$*, for every $n \geq 0$, where* $\det H_{n+r}$ *is given by* (10).

Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence (4) and suppose that the associated nonhomogeneous part $\mathcal{C} = \{c_n\}_{n\geq r}$ is a sequence of type (3) of order s, whose characteristic polynomial is $Q(z) = z^s - b_0 z^{s-1} - b_1 z^{s-2} - \cdots - b_{s-1}$. Let $R(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$ be the characteristic polynomial of the homogeneous part of (4). The linearization process of [2, Theorem 2.1 (Linearization Process)] applied to the sequence (4), allows us to show that $\mathcal{T} = \{v_n\}_{n\geq 0}$ is a sequence of type (3) of order $r+s$, with initial data $v_0, v_1, \ldots, v_{r+s-1}$ and whose coefficients $c_0, c_1, \ldots, c_{r+s}$ are obtained from its characteristic polynomial given by $P(z) = Q(z)R(z)$. Therefore, following Proposition 4.2, we get the following property.

Proposition 4.5. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4) *and $H_{r+n} = (v_{i+j})_{0\leq i,j\leq r+n-1}$ its associated Hankel matrices of order $r+n$. Suppose that $\mathcal{C}$ is a sequence of type* (3) *of order s. Then, we have* $\det H_{r+n} = 0$*, for every $n \geq s$.*

On the other hand, let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$ be a solution of the linear moment problem (1) on a vector on a non-vanishing $x \in \mathcal{H}$, associated with the sequence $\mathcal{T} = \{v_n\}_{n\geq 0}$ defined by (4). By the linear recursive relation (3), related to the linearized expression of (4), we have $\langle A^n P(A)x|x\rangle = \langle P(A)x|A^n x\rangle = 0$, for every $n \geq 0$, where $P(z) = Q(z)R(z)$ is the characteristic polynomial of the linearized sequence of (4). Therefore, we have $\langle A^n P(A)x|A^m P(A)x\rangle = 0$, for every $n \geq 0$, $m \geq 0$, especially $\|A^n P(A)x\| = 0$, for every $n \geq 0$. This implies that $A^n x$ is a linear combination of x, Ax, $\ldots$, $A^{r+s-1}x$. Therefore, when the nonhomogeneous part $\mathcal{C}$ is an $s-GFS$, if the linear moment problem owns a solution A, a self-adjoint operator on a Hilbert space $\mathcal{H}$, then it has a solution A on some $r+s$-dimensional Hilbert space (for more details see [11, Proposition 2.2]). This allows us to suppose that the Hilbert space $\mathcal{H}$ is of finite dimension $(r+s)$. Therefore, we have the following result.

Proposition 4.6. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4)*, with positive definite associated Hankel matrix H_r, and let $P(z)$ the characteristic polynomial of its homogeneous part. Suppose that $\mathcal{C}$ is a linear recursive sequence of type* (3) *of order s, whose characteristic polynomial is $Q(z)$. Then, there exists a* $(\deg(P)+\deg(Q))$*-dimensional Hilbert space $\mathcal{H}_{(\mathcal{T})}$ and a self-adjoint operator A on $\mathcal{H}_{(\mathcal{T})}$, solution of the moment problem* (1).

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Proposition 4.6 shows the main role of the recursiveness of the sequence $\{c_n\}_{n\geq 0}$, in reducing the study of the linear moment problem (1) to the finite dimensional Hilbert space $\mathcal{H}$.

5 Some considerations on the K-moment problems (2) for sequences (4)

The aim here is to apply results of the preceding sections for solving the K-moment problem (2) for nonhomogeneous recursive sequences (4), using results of the linear moments problems in Hilbert spaces $\mathcal{H}$. More precisely, the solution of K-moment problem (2) is obtained in terms of representing measure of the self-adjoint operator A and the vector $x \in \mathcal{H}$ solution of the linear moment problem (1), for the nonhomogeneous recursive sequences (4). The Stieltjes and Hamburger moment problems for the nonhomogeneous recursive sequences (4) are discussed.

5.1 $K-$moment problems associated with sequences (4)

Recall that the purpose of the $K-$moment problem associated with a given sequence $\mathcal{T} = \{v_n\}_{0\leq n\leq p}$, where K is a closed subset of $\mathbb{R}$, is to find a positive Borel measure μ such that Expression (2) is verified, namely,

$$v_n = \int_K t^n d\mu(t) \qquad \text{and} \qquad \operatorname{supp}(\mu) \subset K.$$

As mentioned above, the problem (2) has been studied in the literature, by various methods and techniques. It is called the *full moment problem* when $p = +\infty$ and the *truncated moment problem*, for $p < +\infty$ (see [7–9]). Using the spectral representation of the self-adjoint operators, we can show that the linear moment problem (1) and the moment problem (5.1) are equivalent (see for example [6]). Moreover, using Theorem 4.3 and Theorem 4.4, we get,

Theorem 5.1. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence (4). Suppose that the Hankel matrix $H = (v_{i+j})_{i,j\geq 0}$ is positive. Then, there exists a positive Borel measure μ such that*

$$v_n = \int_K t^n d\mu(t),$$

where $K = \operatorname{supp}(\mu)$. Namely, the there exists a positive Borel measure μ solution of the K-moment problem (2).

Now consider the moment problem (2) for a sequence $\mathcal{T} = \{v_n\}_{n\geq 0}$ given by (4). Let μ be a positive Borel measure of support K. Then, following the proof of Theorem 2.2, we have $v_n = \displaystyle\int_K t^n d\mu(t)$ for every $n \geq 0$, if and only if, $v_n = \displaystyle\int_K t^n d\mu(t)$ for any $n = 0, \ldots, r-1$ and $c_n = \displaystyle\int_K t^{n-r} P(t) d\mu(t)$ for $n \geq r$, where

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$K = \mathrm{supp}(\mu)$. Moreover, a direct computation allows us to get the following result.

Proposition 5.2. *Under the preceding data, the following assertions are equivalent.*

(i) $v_n = \int_K t^n d\mu(t)$*, for every* $n \geq 0$*, where* $K = \mathrm{supp}(\mu)$*.*

(ii) $v_n = \int_K t^n d\mu(t)$ *for* $n = 0, \ldots, 2r-1$ *and* $c_n - \sum_{j=0}^{r-1} a_j c_{n-j-1} = \int_K t^{n-2r} P(t)^2 d\mu(t)$*, for every* $n \geq 2r$*, where* $K = \mathrm{supp}(\mu)$*.*

It is easy to show that the second assertion of the Proposition 5.2 implies that $c_{2k} - \sum_{j=0}^{r-1} a_j c_{2k-j-1} = \int [t^{k-r} P(t)]^2 d\mu(t)$, for any $k \geq r$, and if there exists $k_0 \geq r$ such that $c_{2k_0} - \sum_{j=0}^{r-1} a_j c_{2k_0-j-1} = 0$, then $\mathrm{supp}(\mu) \subset \mathcal{Z}(P) \cup \{0\}$ or equivalently the sequence $\mathcal{T}$ is an $r - GFS$, in which case the sequence $\mathcal{C}$ vanish. This allows us to give a necessary condition for a sequence (4) to be a moment sequences of some positive Borel measure. Thus, we recover Lemma 2.2 of [10], considered for the special case of the Hausdorff moment problem. Since the sequence $\mathcal{C}$ is a nontrivial, if a sequence (4) is a moment sequence of a positive Borel measure μ, we have $c_{2k} > \sum_{j=0}^{r-1} a_j c_{2k-j-1}$, for $k \geq r$. Hence, we can obtain the following.

Proposition 5.3. *Let* $\mathcal{T} = \{v_n\}_{n\geq 0}$ *be a sequence* (4)*. If* $\mathcal{T}$ *is a moment sequences of a positive Borel measure* μ*, then* $c_{2k} > \sum_{j=0}^{r-1} a_j c_{2k-j-1}$ *for any* $k \geq r$*.*

Using Proposition 4.5, we can easily establish the following.

Proposition 5.4. *Let* $\mathcal{T} = \{v_n\}_{n\geq 0}$ *be a sequence* (4)*,* μ *a positive Borel measure and* ρ *a measure given by* $t^r d\rho(t) = P(t) d\mu(t)$*. Then* μ *is a solution of the full moment problem* (2) *associated with* $\mathcal{T}$ *if and only if* μ *is a solution of the truncated moment problem* (2) *associated with* $\mathcal{T}_r = \{v_n\}_{0\leq n\leq r-1}$ *and* $\{c_{n+r}\}_{n\geq 0}$ *is a moment sequences of* ρ*.*

Particularly, when $\mathcal{T} = \{v_n\}_{n\geq 0}$ is a sequence of type (3) of order r (i.e $c_n = 0$, for every $n \geq 0$), then the second assertion of the preceding proposition is equivalent to the fact that μ is a solution of the truncated moment problem (2) associated with $\mathcal{T}_r = \{v_n\}_{0\leq n\leq r-1}$ and $\int_K t^n d\rho(t) = \int_K t^{n-r} P(t) d\mu(t)$, for every $n \geq r$. The last statement is equivalent to $\mathrm{supp}(\mu) \subset \mathcal{Z}(P)$, and we obtain Lemma 2.2 of [10] in the particular case of the Hausdorff moment problem.

5.2 Moment problems (2) associated with sequences (4), with c_n satisfying (3)

Let consider the linear moment problem (1) for sequence sequences (4), where the sequence $\mathcal{C} = \{c_n\}_{n\geq r}$ satisfies the linear recursive relation (3). Then, by

Proposition 4.2, Theorem 4.4, Proposition 4.6 and Theorem 5.1, we get the following result concerning the Hamburger moment problem for sequences (4).

Theorem 5.5. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4)*. Suppose that $\mathcal{C} = \{c_n\}_{n\geq 0}$ is a sequence of type* (3) *of order s. Then, a necessary and sufficient condition that there exists a measure μ solution of the truncated Hamburger moment problem associated with a sequence $\mathcal{T} = \{v_n\}_{n\geq 0}$ is that the Hankel matrix H_{r+s} is positive definite or equivalently* $\det H_n > 0$ *for $n = 0, 1, \ldots, r+s$.*

Similarly, we get the following result concerning the Stieltjes moment problem for sequences (4).

Theorem 5.6. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4)*. Suppose that $\mathcal{C} = \{c_n\}_{n\geq 0}$ is a sequence of type* (3) *of order s. Then, a necessary and sufficient condition that there exists a measure μ solution of the truncated Stieltjes moment problem associated with a sequence $\mathcal{T} = \{v_n\}_{n\geq 0}$ is that the two matrices H_{r+s} and S_{r+s} are positive definite or equivalently* $\det H_n > 0$ *and* $\det S_n > 0$ *for $n = 0, 1, \ldots, r+s$.*

Note that a similar result can be established for the Hausdorff moment problem.

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The Atomic Solution for Fractional Wave Type Equation

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ABSTRACT

Sometimes, it is not possible to find a general solution for some differential equations using some classical methods, like separation of variables. In such a case, one can try to use theory of tensor product of Banach spaces to find certain solutions, called atomic solution. The aim of this paper is to find atomic solution for conformable non-linear wave equation.

Key Words: fractional wave type equation; conformable derivative; atomic solution.

1 Introduction

In [Khalil et al., 2014], a new definition called α-conformable fractional derivative was introduced as follows:
Letting $\alpha \in (0,1)$, and $f : E \subseteq (0,\infty)$. Then for $x \in E$

$$D^\alpha f(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}. \tag{1}$$

If the limit exists then it is called the α-conformable fractional derivative of f at x.
For x=0, if f is α-differentiable on $(0,r)$ for some $r > 0$, and $\lim_{x\to 0} D^\alpha f(0)$ exists then we define $D^\alpha f(0) = \lim_{x\to 0} D^\alpha f(0)$. The new definition satisfies:

1. $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in R$.

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2. $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

Further, for $\alpha \in (0, 1]$ and f,g are α-differentiable at a point t, with $g(t) \neq 0$. Then

1. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
2. $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}, g(t) \neq 0$.

We list here the fractional derivatives of certain functions,

1. $T_\alpha(t^p) = pt^{p-\alpha}$.
2. $T_\alpha(\sin \frac{1}{\alpha}t^\alpha) = \cos \frac{1}{\alpha}t^\alpha$.
3. $T_\alpha(\cos \frac{1}{\alpha}t^\alpha) = -\sin \frac{1}{\alpha}t^\alpha$.
4. $T_\alpha e^{\frac{1}{\alpha}t^\alpha} = e^{\frac{1}{\alpha}t^\alpha}$.

On letting $\alpha = 1$ in these derivatives, we get the corresponding classical rules for the ordinary derivatives.
One should notice that a function could be α-conformable differentiable at a point but not differentiable, for example, take $f(t) = 2\sqrt{t}$. Then $T_{\frac{1}{2}}(f)(0) = 1$.
This is not the case for the known classical fractional derivatives, since $T_1(f)(0)$ does not exist.
A vast number of researcher dedicated so much of their work to study conformable derivatives and its applications. Among them, [Abdeljawad, 2015], [Abu Hammad and Khalil, 2014], [Aldarawi, 2018], [Alhabees and Aldarawi, 2020], [ALHabees, 2021], [ALHorani and Khalil, 2018], [Anderson et al., 2018], [Atangana et al., 2015], [Chung, 2015], [Hammad and Khalil, 2014], [Khalil et al., 2016], [Kilbas,], [Mhailan et al., 2020].

2 Atomic Solution

Let X and Y be two Banach spaces and X^* be the dual of X. Assume $x \in X$ and $y \in Y$. The operator $T : X^* \to Y$, defined by

$$T(x^*) = x^*(x)y \tag{2}$$

is bounded one rank linear operator. We write $x \otimes y$ for T. Such operators are called atoms. Atoms are among the main ingredient in the theory of tensor products.

Atoms are used in theory of best approximation in Banach spaces, see [Al Horani et al., 2016]. According to [Khalil, 1985], one of the known results that we need in our paper is: if the sum of two atoms is an atom, then either the first components are dependent or the second are dependent.
For more on tensor product of Banach spaces we refer to [Deeb and Khalil, 1988] and [Khalil, 1985].
Our main object in this paper is to find an atomic solution of the equation

$$D_t^\alpha D_t^\alpha u = c^2 D_x^\beta D_x^\beta u + D_t^\alpha D_x^\beta u. \tag{3}$$

This is called the conformable non-linear wave equation, where c is constant. Let $c = 1$ for simplicity to get

$$D_t^\alpha D_t^\alpha u = D_x^\beta D_x^\beta u + D_t^\alpha D_x^\beta u. \tag{4}$$

If one tries to solve this equation via separation of variables, then it is not possible since the variables can not be separated.

3 Procedure

Let $u(x,t) = X(x)T(t)$. substitute in equation (4) to get:

$$X(x)T^{2\alpha}(t) = X^{2\beta}(x)T(t) + X^\beta(x)T^\alpha(t). \tag{5}$$

This can be written in tensor product form as:

$$X(x) \otimes T^{2\alpha}(t) = X^{2\beta}(x) \otimes T(t) + X^\beta(x) \otimes T^\alpha(t). \tag{6}$$

Let us consider the following conditions: $X(0) = 1$, $X^\beta(0) = 1$.
In equation (6), we have the situation: the sum of two atoms is an atom. Hence, we have two cases:

3.1 case I: $X^{2\beta}(x) = X^\beta(x)$

The situation of case I: $X^{2\beta}(x) = X^\beta(x)$, using the result in [Al-Horani et al., 2020], we get

$$X(x) = e^{\frac{x^\beta}{\beta}}. \tag{7}$$

Now, we substitute in (6) to get

3

$$
\begin{aligned}
e^{\frac{x^\beta}{\beta}} \otimes T^{2\alpha}(t) &= e^{\frac{x^\beta}{\beta}} \otimes T(t) + e^{\frac{x^\beta}{\beta}} \otimes T^{\alpha}(t). \\
e^{\frac{x^\beta}{\beta}} \otimes T^{2\alpha}(t) &= e^{\frac{x^\beta}{\beta}} \otimes [T(t) + T^{\alpha}(t)]. \\
T^{2\alpha}(t) &= T(t) + T^{\alpha}(t). \qquad (8)
\end{aligned}
$$

Hence, $T^{2\alpha}(t) = T(t)+T^{\alpha}(t)$. Again, using the result in [Al-Horani et al., 2020],

$$T(t) = c_1 e^{(\frac{1+\sqrt{5}}{2})\frac{t^\alpha}{\alpha}} + c_2 e^{(\frac{1-\sqrt{5}}{2})\frac{t^\alpha}{\alpha}}. \qquad (9)$$

Using the conditions $T(0) = T^{\alpha}(0) = 1$, we get

$$T(t) = \frac{\sqrt{5}+1}{2\sqrt{5}} e^{(\frac{1+\sqrt{5}}{2})\frac{t^\alpha}{\alpha}} + \frac{\sqrt{5}-1}{2\sqrt{5}} e^{(\frac{\sqrt{5}-1}{2})\frac{t^\alpha}{\alpha}}. \qquad (10)$$

From (7) and (10), we obtain the atomic solution of (4) as follows:

$$u(x,t) = e^{\frac{x^\beta}{\beta}} \left(\frac{\sqrt{5}+1}{2\sqrt{5}} e^{(\frac{1+\sqrt{5}}{2})\frac{t^\alpha}{\alpha}} + \frac{\sqrt{5}-1}{2\sqrt{5}} e^{(\frac{1-\sqrt{5}}{2})\frac{t^\alpha}{\alpha}} \right). \qquad (11)$$

3.2 case II: $T(t) = T^{\alpha}(t)$

This is conformable linear differential equation. Hence, we can use the result in [Khalil, 1985], or use the fact that

$$T^{\alpha}(t) = t^{1-\alpha} T\prime(t). \qquad (12)$$

To get

$$
\begin{aligned}
T(t) &= t^{1-\alpha} T\prime(t) \\
\frac{dT(t)}{T(t)} &= t^{\alpha-1} dt \\
LnT(t) &= \frac{t^\alpha}{\alpha} + k. \qquad (13)
\end{aligned}
$$

Where k is constant. Hence,

$$T(t) = Ke^{\frac{t^\alpha}{\alpha}}, K = e^k. \qquad (14)$$

Again, by using the conditions $T(0) = T^{\alpha}(0) = 1$, we get

$$T(t) = e^{\frac{t^\alpha}{\alpha}}. \qquad (15)$$

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Substitute in equation (4) to get

$$\begin{aligned} X(x) \otimes e^{\frac{t^\alpha}{\alpha}} &= (X^{2\beta}(x) + X^\beta(x)) \otimes e^{\frac{t^\alpha}{\alpha}} \\ X(x) &= X^{2\beta}(x) + X^\beta(x). \end{aligned} \tag{16}$$

Again, by using the result in [Khalil, 1985], and the conditions $X(0) = X^\beta(0) = 1$, we get

$$X(x) = \left(\frac{3+\sqrt{5}}{2\sqrt{5}}\right) e^{\frac{-1+\sqrt{5}}{2}\frac{x^\beta}{\beta}} + \left(\frac{-3+\sqrt{5}}{2\sqrt{5}}\right) e^{\frac{-1-\sqrt{5}}{2}\frac{x^\beta}{\beta}} \tag{17}$$

From (15) and (17), we obtain the atomic solution of (4) as follows:

$$u(x,t) = \left(\left(\frac{3+\sqrt{5}}{2\sqrt{5}}\right) e^{\frac{-1+\sqrt{5}}{2}\frac{x^\beta}{\beta}} + \left(\frac{-3+\sqrt{5}}{2\sqrt{5}}\right) e^{\frac{-1-\sqrt{5}}{2}\frac{x^\beta}{\beta}}\right) e^{\frac{t^\alpha}{\alpha}} \tag{18}$$

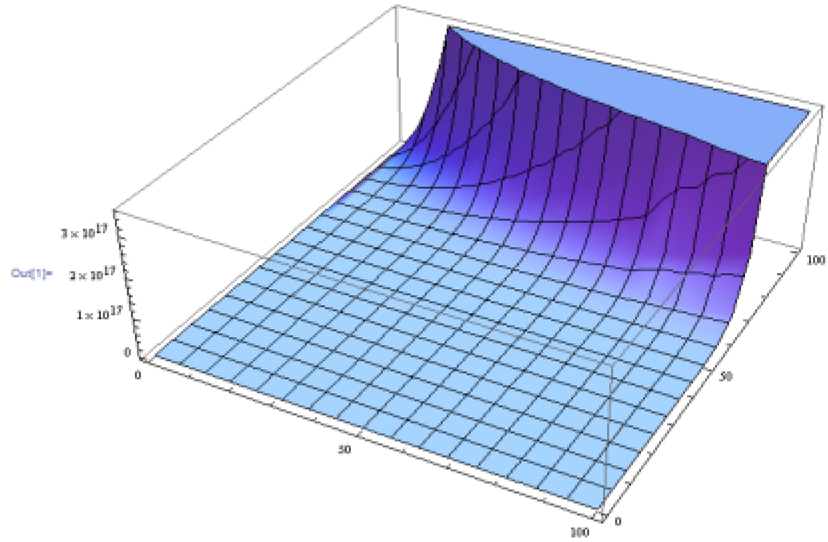
3.3 Example

Considering the following fractional wave equation

$$D_t^{0.5} D_t^{0.5} u = D_x^{0.2} D_x^{0.2} u + D_t^{0.5} D_x^{0.2} u. \tag{19}$$

The solution of (19) is

$$u(x,t) = e^{\frac{x^{0.2}}{0.2}} \left(\frac{\sqrt{5}+1}{2\sqrt{5}} e^{(\frac{1+\sqrt{5}}{2})\frac{t^{0.5}}{0.5}} + \frac{\sqrt{5}-1}{2\sqrt{5}} e^{(\frac{1-\sqrt{5}}{2})\frac{t^{0.5}}{0.5}} \right). \tag{20}$$



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Asymptotic behavior of solutions of a class of time-varying systems with periodic perturbation

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Abstract This paper deals with stability of nonlinear differential equations with parameter with periodic perturbation. We determine values of the parameter under which the solutions of the perturbed systems could be uniformly exponentially stable. Sufficient conditions for global uniform asymptotic stability and/or practical stability in terms of Lyapunov-like functions are obtained in the sense that the trajectories converge to a small ball centered at the origin. Moreover, to illustrate the applicability of our result, we study the stabilization problem for a class of control system.

Keywords: Differential equations, parametric systems, perturbation, asymptotic behavior of solutions.

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1 Introduction

The investigation of stability analysis of nonlinear uncertain systems is an important topic in systems theory. The problem of stability analysis of nonlinear time-varying systems has attracted the attention of several researchers and has produced a vast body of important results (see [2]-[26], [29], [32], [33], [34] and the references therein). There have been a number of interesting developments in searching the stability criteria for nonlinear differential systems, but most have been restricted to finding the asymptotic stability conditions for some classes of certain systems. In particular, parametric stability for nonlinear systems is an interesting area of research, and it naturally arises in diverse fields such as population biology, economics, neural networks, and chemical processes.

Basically, parametric stability for nonlinear systems addresses the stability of equilibria for nonlinear systems with real parametric uncertainty, especially the feasibility of equilibria and the stability nature of the equilibria with respect to small variations of the real parametric uncertainty (see [25]). Dynamic systems governed by ordinary differential equations with periodically varying coefficients have been studied since one and a half centuries ago (see [12], [14], [19] and the references therein).
Mathieu [31] introduced a differential equation with periodic coefficient and Hill [24] presented the first ever solution technique of linear periodic equations. Lyapunov [30] demonstrated the Lyapunov-Floquet transformation for autonomous systems which is a linear periodic system into a dynamically equivalent time-invariant form. Unlike the differential systems without parameters, studying stability of differential parametric systems with periodic coefficients may not be easily verified ([16]-[17]).
It is well known that for linear parametric systems of the form: $\dot{x} = A(\alpha)x$, α is a real parameter which can be constant or depending on time. For technical reasons, it is important to distinguish between constant and time-varying parameters. Constant parameters have a fixed value that is known only approximately. In this case, the underlying dynamical linear system is time invariant. Time-varying parameter $\alpha(t)$ is a certain function which varies in some range and the resulting system is then time-varying. Kharitonov's theorem (see [27]) gives a simple necessary and sufficient condition for parametric system where a quadratic Lyapunov function is used to solve the problem of stability. Barmish in [3] introduced the notion of parameter dependent Lyapunov functions for continuous-time linear systems whose dynamic matrices are affected by bounded uncertain time-varying parameters. Floquet [20] developed the complete study for stability of linear time-periodic differential equations. Based on Floquet theory the stability of the linear system with time-periodic coefficients can be determined from the eigenvalues of a certain matrix. These eigenvalues are often called Floquet multipliers. He proved that, if all Floquet multipliers have magnitude less than one, the linear system with time-periodic coefficient is asymptotically stable. In general to solve the problem of stability the usual techniques are related to some linear matrices inequalities that finding an adequate Lyapunov matrix to solve a system of Lyapunov inequalities which is a convex program. Perturbation theory is a pertinent discipline for the applications of time parametric dynamics which is a compilation of methods systematically used to evaluate the global behavior of solutions to differential equations. This motivates us to study the problem of uniform exponential stability of perturbed systems by assuming that the nominal associated system is globally uniformly asymptotically stable by imposing some restrictions on the size of perturbations in particular that are periodic in time.
The goal is to obtain estimates for the solutions of perturbed differential equations and to get uniform boundedness and uniform convergence to a small neighborhood of the origin. The notion of practical stability, (see [6]), is introduced in a special case. We determine values of parameters under which the systems are uniformly practically exponentially stable where some estimates on the decay rate of solutions at infinity are obtained. Finally, we give an application for the stabilization a class of control parametric system.

2 General definitions

Consider the non-autonomous system

$$\frac{dx}{dt} = f(t, x) \tag{1}$$

where $f : [0, \infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous in t and locally Lipschitz in x on $[0, \infty) \times \mathbb{R}^n$. The origin is an equilibrium point for (1), if $f(t, 0) = 0, \quad \forall t \geq 0$.

Definition 1. ***(Exponential stability)*** *The zero solution of system (1) is exponentially stable if there exist positive constants c, μ, and λ such that*

$$\|x(t)\| \leq \mu \|x(t_0)\| e^{-\lambda(t-t_0)}, \quad \forall \ \|x(t_0)\| < c \tag{2}$$

and globally exponentially stable if (2) *is satisfied for any initial state $x(t_0) \in \mathbb{R}^n$.*

The exponential stability is more important than stability, also the desired system may be unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable, in particular when $f(t, 0) \neq 0$, thus the notion of *practical stability* is more suitable in several situations than Lyapunov stability, it means that the trajectories converge to a small neighborhood of the origin, in the sense of uniform stability and uniform attractivity of system (1) with respect a certain ball $B_r = \{x \in \mathbb{R}^n / \|x\| \leq r\}$.

Definition 2. ***(Uniform stability of*** B_r***)*** *B_r is uniformly stable if for all $\varepsilon > r$, there exists $\delta = \delta(\varepsilon) > 0$, such that*

$$\|x(t_0)\| < \delta \Longrightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0. \tag{3}$$

Definition 3. ***(Uniform attractivity of*** B_r***)*** *B_r is uniformly attractive, if for $\varepsilon > r$, $t_0 > 0$ and $x(t_0) \in D$, there exists $T(\varepsilon, x(t_0)) > 0$, such that*

$$\|x(t)\| < \varepsilon, \quad \forall t \geq t_0 + T(\varepsilon, x(t_0)). \tag{4}$$

B_r is globally uniformly attractive if (4) *is satisfied for all $x(t_0) \in \mathbb{R}^n$.*

Definition 4. ***(Practical stability)*** *System (1) is said uniformly practically asymptotically stable, if there exists $B_r \subset \mathbb{R}^n$, such that B_r is uniformly stable and uniformly attractive. It is globally uniformly practically asymptotically stable if $x(t_0) \in \mathbb{R}^n$.*

Definition 5. *System (1) is said uniformly exponentially convergent to B_r, if there exist $\gamma > 0$ and $k \geq 0$, such that*

$$\|x(t)\| \leq k\|x(t_0)\| \exp(-\gamma(t - t_0)) + r, \quad \forall t \geq t_0, \quad \forall x(t_0) \in \mathbb{R}^n. \tag{5}$$

If $x(t_0) \in \mathbb{R}^n$, the system is globally uniformly exponentially convergent to B_r.
We say that the system is globally uniformly practically exponentially stable if for $r > 0$, it is globally uniformly exponentially convergent to B_r.

Here, we study the asymptotic behavior of a small ball centered at the origin for $0 \leq \| \ x(t) \ \| -r$, so that if $r = 0$ we find the classical definition of the uniform asymptotic or exponential stability of the origin viewed as an equilibrium point.

3 Problem formulation

We consider the following system of differential equations

$$\frac{dx}{dt} = \mu(A(\alpha(t)) + B(t))x + \nu\varphi(t,x), \quad t \geq 0, \tag{6}$$

where $A(\alpha(t)) \in \mathbb{R}^{n\times n}$ is a matrix given by $A(\alpha(t)) = \alpha_1(t)A_1 + \alpha_2(t)A_2$, with $\alpha_1(t) + \alpha_2(t) = 1, \alpha_i(t) \in \mathbb{R}^+,\ \forall t \geq 0,\ B(t) \in \mathbb{R}^{n\times n}$ is T-periodic matrix, $\mu, \nu \in \mathbb{R}$ are parameters and $\varphi(t,x)$ is a smooth vector function such that, for all $t \geq 0$ and $x \in \mathbb{R}^n$

$$\varphi(t+T,x) = \varphi(t,x)$$

and

$$\|\varphi(t,x)\| \leq k\|x\|^{1+\delta} + r, \quad \delta \geq 0,\ k > 0,\ r > 0. \tag{7}$$

Suppose that the spectrum of matrices A_1 and A_2 belong to the left half-plane $\{\lambda \in \mathbb{C}, \mathcal{R}(\lambda) < 0\}$ and

$$\int_0^T B(t)d(t) = 0. \tag{8}$$

Throughout this paper, we indicate the following domains:

$$I_1 = \{\mu \in \mathbb{R}, 0 < \mu < \mu_0\}, \qquad I_2 = \{\nu \in \mathbb{R}, |\nu| < \nu_0\},$$

such that the system (6) is practically uniformly exponentially stable for $\mu \in I_1$, $\nu \in I_2$. Moreover, we obtain estimates on the solutions of (6) that guarantee exponential decay when $t \longrightarrow +\infty$ to a certain ball $B(0,r_i)$ with a radius $r_i, i = 1,2$.

Remark For $\mu = \nu = 1$, the system (6) can be seen as a perturbed system (see [8], [9]).

Notations: The following notations will be used throughout this paper. For a matrix X, the notation X^* denotes the transpose of matrix X. $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ denote the minimum and the maximum eigenvalues of X respectively.

Since

$$spect(A_i)_{i=1,2} \subset \{\lambda \in \mathbb{C}, Re(\lambda) < 0\},$$

then, there exist symmetric and positive definite matrices H_1 and H_2 solutions of the matrices Lyapunov equations (see [26] for the existence and uniqueness of the matrices $H_i,\ i = 1,2$),

$$H_1A_1 + A_1^*H_1 = -I \tag{9}$$

and

$$H_2A_2 + A_2^*H_2 = -I. \tag{10}$$

The matrices $H_i,\ i = 1,2$ satisfy:

$$H_i = \int_0^\infty e^{sA_i^*}e^{sA_i}ds.$$

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In many cases, it is hard to find a common positive-definite matrix $H = H_1 = H_2$. In fact, the existence of a common positive-definite matrix depends on the difference of the two matrices $A_i, i = 1, 2$. In order to solve these problems, many scholars have made many further investigations. For example, in [28], the authors showed that, if the matrices A_1 and A_2 are real Hurwitz matrices, and that their difference is rank one, then A_1 and A_2 have a common quadratic Lyapunov function if and only if the product $A_1 A_2$ has no real negative eigenvalue. We can solve this problem, in the special case when $A_1 + A_1^* = A_2 + A_2^*$, we get

$$H = \int_0^\infty e^{sA_1^*} e^{sA_1} ds = \int_0^\infty e^{sA_2^*} e^{sA_2} ds.$$

To facilitate our task, we will suppose that, (9) and (10) have a unique solution $H = H^* > 0$.
We have

$$\gamma_1 \|x\|^2 \leq \langle Hx, x\rangle \leq \|H\| \|x\|^2,$$

where $\gamma_1 = \lambda_{\min}(H)$.

Now, In order to study the asymptotic behavior of solutions, we shall impose some conditions on the parameters under which the system (6) can be practically uniformly exponentially stable.

Theorem 1. *Let*

$$\beta_1 = \max_{\tau\in[t_0,t_0+T]} \|H \int_{t_0}^{\tau} B(s)ds + \int_{t_0}^{\tau} B^*(s)dsH\|,$$

$$\beta_2 = \max_{\tau\in[t_0,t_0+T]} \|(H \int_{t_0}^{\tau} B(s)ds + \int_{t_0}^{\tau} B^*(s)dsH)(A_1 + B(\tau))\|,$$

$$\beta_3 = \max_{\tau\in[t_0,t_0+T]} \|(H \int_{t_0}^{\tau} B(s)ds + \int_{t_0}^{\tau} B^*(s)dsH)(A_2 + B(\tau))\|,$$

and

$$\mu_0 = \min\{\frac{\gamma_1}{\beta_1}, \frac{1}{2\beta}\} \quad \textit{where } \beta = \max\{\beta_2, \beta_3\}.$$

Let H be a solution to the matrices Lyapunov equations (9) and (10) and $\delta = 0$. Then, for parameters μ and ν such that

$$0 < \mu < \mu_0 \qquad \textit{and} \quad 2\mu\beta + 2|\nu|\ k\left(\frac{\|H\|}{\mu} + \beta_1\right) < 1,$$

and for any initial data $x(t_0) \in \mathbb{R}^n$, the solutions of system (6) converge exponentially towards the ball $B(0, r_1)$ whose radius is given by

$$r_1 = 2|\nu| r \frac{(\frac{\|H\|}{\mu} + \beta_1)^2}{(\frac{\gamma_1}{\mu} - \beta_1)\left(1 - 2\mu\beta - 2|\nu| k(\frac{\|H\|}{\mu} + \beta_1)\right)}.$$

Remark Note that, if $\nu = \nu(t)$ with $|\nu(t)| \longrightarrow 0$ as $t \longrightarrow +\infty$, then the solution of system (6) tend to zero when t tends to infinity.

Proof Define the following matrix

$$H(t,\mu) = \frac{1}{\mu}H - H\int_{t_0}^{t} B(s)ds - \int_{t_0}^{t} B^*(s)ds\ H. \tag{11}$$

Since $H = H^*$, it follows that

$$H(t,\mu) = H^*(t,\mu)$$

and by (8), the matrix $H(t,\mu)$ is T-periodic, i.e.

$$H(t+T,\mu) = H(t,\mu).$$

Let $x(t)$ be a solution to (6), then the function

$$h(t,\mu,\nu) = \langle H(t,\mu)x(t), x(t)\rangle$$

is continuously differentiable on t. It follows that, the derivative of $h(t,\mu,\nu)$ is given by

$$\frac{d}{dt}h(t,\mu,\nu) = \langle \frac{d}{dt}H(t,\mu)x(t), x(t)\rangle + \langle H(t,\mu)\frac{d}{dt}x(t), x(t)\rangle + \langle H(t,\mu)x(t), \frac{d}{dt}x(t)\rangle.$$

Since

$$\frac{d}{dt}H(t,\mu) = -HB(t) - B^*(t)H,$$

then

$$\begin{aligned}
\frac{d}{dt}h(t,\mu,\nu) \quad = \quad & -\langle (HB(t) + B^*(t)H)x(t), x(t)\rangle \\
& +\langle \mu H(t,\mu)(A(\alpha(t)) + B(t))x(t), x(t)\rangle \\
& +\langle \mu(A(\alpha(t))^* + B^*(t))H(t,\mu)x(t), x(t)\rangle \\
& +\nu\langle H(t,\mu)\varphi(t,x), x(t)\rangle + \nu\langle H(t,\mu)x(t), \varphi(t,x)\rangle.
\end{aligned}$$

Using the definition of matrix $H(t,\mu)$, we obtain

$$\begin{aligned}
\frac{d}{dt}h(t,\mu,\nu) \quad = \quad & \langle (-HB(t) - B^*(t)H)x(t), x(t)\rangle + \langle H(A(\alpha(t)) + B(t))x(t), x(t)\rangle \\
& -\mu\langle (H\int_{t_0}^{t} B(s)ds + \int_{t_0}^{t} B^*(s)ds\ H)(A(\alpha(t)) + B(t))x(t), x(t)\rangle \\
& +\langle (A(\alpha(t))^* + B^*(t))Hx(t), x(t)\rangle \\
& -\mu\langle (A(\alpha(t))^* + B^*(t))(H\int_{t_0}^{t} B(s)ds + \int_{t_0}^{t} B^*(s)ds\ H)x(t), x(t)\rangle \\
& +2\left(\nu\langle H(t,\mu)\varphi(t,x), x(t)\rangle\right).
\end{aligned}$$

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Replacing $A(\alpha(t))$ by its value and multiplying $B(t)$ by $(\alpha_1(t)+\alpha_2(t))$, we get

$$\begin{aligned}
\frac{d}{dt}h(t,\mu,\nu) \;=\;& \langle \alpha_1(t)(HA_1 + A_1^*H) + \alpha_2(t)(HA_2 + A_2^*H)x(t), x(t)\rangle \\
& -\alpha_1(t)\mu\Big(\langle (H\int_{t_0}^t B(s)ds + \int_{t_0}^t B^*(s)ds\ H)(A_1+B(t))x(t), x(t)\rangle \\
& + \langle (A_1+B(t))^*(H\int_{t_0}^t B(s)ds + \int_{t_0}^t B^*(s)ds\ H)x(t), x(t)\rangle\Big) \\
& -\alpha_2(t)\mu\Big(\langle (H\int_{t_0}^t B(s)ds + \int_{t_0}^t B^*(s)ds\ H)(A_2+B(t))x(t), x(t)\rangle \\
& + \langle (A_2+B(t))^*(H\int_{t_0}^t B(s)ds + \int_{t_0}^t B^*(s)ds\ H)x(t), x(t)\rangle\Big) \\
& +2\left(\nu\langle H(t,\mu)\varphi(t,x), x(t)\rangle\right).
\end{aligned} \tag{12}$$

Taking into account (9) and (10) and using the fact that $0<\mu<\mu_0$, we obtain the following estimate

$$\begin{aligned}
\frac{d}{dt}h(t,\mu,\nu) \;\le\;& -\|x(t)\|^2 \\
& +2\mu\alpha_1(t)\max_{\tau\in[t_0,t_0+T]}\|(H\int_{t_0}^\tau B(s)ds + \int_{t_0}^\tau B^*(s)dsH)(A_1+B(\tau))\|\|x(t)\|^2 \\
& +2\mu\alpha_2(t)\max_{\tau\in[t_0,t_0+T]}\|(H\int_{t_0}^\tau B(s)ds + \int_{t_0}^\tau B^*(s)dsH)(A_2+B(\tau))\|\|x(t)\|^2 \\
& +2|\nu|k\left(\frac{\|H\|}{\mu}+\beta_1\right)\|x(t)\|^2 + 2|\nu|r\left(\frac{\|H\|}{\mu}+\beta_1\right)\|x(t)\| \\
\le\;& -\left(1-2\mu\beta-2|\nu|\ k\left(\frac{\|H\|}{\mu}+\beta_1\right)\right)\|x(t)\|^2 \\
& +2|\nu|r\left(\frac{\|H\|}{\mu}+\beta_1\right)\|x(t)\|.
\end{aligned}$$

Since the matrix $H(t,\mu)$ is positive definite for $0<\mu<\mu_0$, it follows that

$$0 < (\frac{1}{\mu}\gamma_1-\beta_1)I \le H(t,\mu) \le (\frac{1}{\mu}\|H\|+\beta_1)I.$$

Thus,

$$\begin{aligned}
\frac{d}{dt}h(t,\mu,\nu) \;\le\;& -\frac{1-2\mu\beta-2|\nu|\ k\left(\frac{\|H\|}{\mu}+\beta_1\right)}{\frac{1}{\mu}\|H\|+\beta_1}h(t,\mu,\nu) \\
+\;& 2|\nu|r\frac{(\frac{\|H\|}{\mu}+\beta_1)}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}}\sqrt{h(t,\mu,\nu)}.
\end{aligned}$$

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Let $\mathcal{H}(t,\mu,\nu)=\sqrt{h(t,\mu,\nu)}$, it follows that,

$$\begin{aligned}\frac{d}{dt}\mathcal{H}(t,\mu,\nu) &\leq -\frac{1-2\mu\beta-2|\nu|\ k\left(\frac{\|H\|}{\mu}+\beta_1\right)}{2(\frac{\|H\|}{\mu}+\beta_1)}\mathcal{H}(t,\mu,\nu)\\ &+ |\nu|r\frac{\frac{\|H\|}{\mu}+\beta_1}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}}\end{aligned}$$

which implies that

$$\begin{aligned}\mathcal{H}(t,\mu,\nu) &\leq \mathcal{H}(t_0,\mu,\nu)\exp\left(-\frac{1-2\mu\beta-2|\nu|\ k\left(\frac{\|H\|}{\mu}+\beta_1\right)}{2(\|H\|+\mu\beta_1)}\mu(t-t_0)\right)\\ &+2|\nu|r\frac{(\frac{\|H\|}{\mu}+\beta_1)^2}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}\left(1-2\mu\beta-2|\nu|k(\frac{\|H\|}{\mu}+\beta_1)\right)}\\ &\leq \sqrt{\frac{\|H\|}{\mu}}\|x(t_0)\|\exp\left(-\frac{1-2\mu\beta-2|\nu|\ k\left(\frac{\|H\|}{\mu}+\beta_1\right)}{2(\|H\|+\mu\beta_1)}\mu(t-t_0)\right)\\ &+2|\nu|r\frac{(\frac{\|H\|}{\mu}+\beta_1)^2}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}\left(1-2\mu\beta-2|\nu|k(\frac{\|H\|}{\mu}+\beta_1)\right)}\end{aligned}$$

and consequently

$$\begin{aligned}\|x(t)\| &\leq \sqrt{\frac{\|H\|}{\gamma_1-\mu\beta_1}}\exp\left(-\frac{1-2\mu\beta-2|\nu|\ k\left(\frac{\|H\|}{\mu}+\beta_1\right)}{2(\|H\|+\mu\beta_1)}\ \mu(t-t_0)\right)\|x(t_0)\|\\ &+ 2|\nu|r\frac{(\frac{\|H\|}{\mu}+\beta_1)^2}{(\frac{\gamma_1}{\mu}-\beta_1)\left(1-2\mu\beta-2|\nu|k(\frac{\|H\|}{\mu}+\beta_1)\right)}.\end{aligned}$$

Thus, we obtain an estimation as in Definition 5. Hence, the solutions of system (6) converge exponentially towards the ball $B(0,r_1)$ whose radius is given by

$$r_1=2|\nu|r\frac{(\frac{\|H\|}{\mu}+\beta_1)^2}{(\frac{\gamma_1}{\mu}-\beta_1)\left(1-2\mu\beta-2|\nu|k(\frac{\|H\|}{\mu}+\beta_1)\right)}.$$

Remark A simple verification shows that $r_1>0$.

In the next part of this paper, a new class of functions appears: functions that depend on a set of constant parameters, that is, $f=f(t,x,\varepsilon)$, where $\varepsilon\in\mathbb{R}^p$. The constant parameters could represent physical parameters of the system and the study of perturbation of these parameters accounts for modeling errors or changes in the parameter values due to aging. Let begin by introducing the following lemma.

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Lemma (see [26]) Let $f(t,x,\varepsilon)$ be continuous in (t,x,ε) and locally Lipschitz in x (uniformly in t and ε) on $[t_0,+\infty[\times\mathbb{R}^n\times\{\|\varepsilon-\varepsilon_0\|\leq c\}$. Let $y(t,\varepsilon_0)$ be a solution of $\dot{x}=f(t,x,\varepsilon_0)$ with $y(t_0,\varepsilon_0)=y_0\in\mathbb{R}^n$. Suppose $y(t,\varepsilon_0)$ is defined and belongs to $\mathbb{R}^n$ for all $t\geq t_0$. Then, given $\lambda>0$, there is $\gamma>0$ such that, if

$$\|z_0-y_0\|<\gamma \textit{ and } \|\varepsilon-\varepsilon_0\|<\gamma$$

then there is a unique solution $z(t,\varepsilon)$ of $\dot{x}=f(t,x,\varepsilon)$ defined for $t\geq t_0$, with $z(t_0,\varepsilon)=z_0$, and $z(t,\varepsilon)$ satisfies

$$\|z(t,\varepsilon)-y(t,\varepsilon_0)\|<\lambda,\quad \forall t\geq t_0.$$

Quite often when we study the state equation $\dot{x}=f(t,x,\varepsilon)$, where $\varepsilon\in\mathbb{R}^p$, we need to compute bounds on the solution $x(t)$ without computing the solution itself. That is why, in order to make our tache more easy, we will solve the differential equation $\dot{x}=f(t,x,\varepsilon_0)$ where ε_0 is a parameter sufficiently close to ε, i.e., $\|\varepsilon-\varepsilon_0\|$ sufficiently small and after that we will approximate the solution of $\dot{x}=f(t,x,\varepsilon)$.

Theorem 2. *Let H be a solution to the matrices Lyapunov equations (9) and (10). Let β_1, β_2, β_3, β and μ_0 be defined in the Theorem 1, let $\delta>0$, $\rho>0$ and*

$$\nu_0=\frac{\mu^{1-\delta/2}\,(\gamma_1-\mu\beta_1)^{1+\delta/2}\,(1-2\mu\beta)}{2\;k(\|H\|+\mu\beta_1)^2\,(\sqrt{\frac{\|H\|}{\mu}}\rho+\gamma)^\delta}$$

with γ is some constant. Then, for $0<\mu<\mu_0$, $|\nu|<\nu_0$ and for any initial data

$$x(t_0)\in\mathbb{R}^n,\quad \|x(t_0)\|\leq\rho,$$

the system (6) is practically uniformly exponentially stable.

Proof Let $x(t)$ be a solution to system (6) and $H(t,\mu)$ be defined by (11). From the proof of Theorem 1, the function $h(t,\mu,\nu)$ satisfy the inequality (12). By the definition of matrix $H(t,\mu)$ and taking into account that $\|\varphi(t,x)\|\leq k\|x\|^{1+\delta}+r$, we obtain the following estimate

$$\begin{aligned}\frac{d}{dt}h(t,\mu,\nu) &\leq -(1-2\mu\beta)\|x(t)\|^2+2|\nu|k\left(\frac{\|H\|}{\mu}+\beta_1\right)\|x(t)\|^{2+\delta}\\ &\quad +2|\nu|r\left(\frac{\|H\|}{\mu}+\beta_1\right)\|x(t)\|.\end{aligned}$$

Since

$$\|x(t)\|^2\leq\frac{h(t,\mu,\nu)}{(\frac{1}{\mu}\gamma_1-\beta_1)}\quad \textit{and}\quad \|x(t)\|^\delta\leq\frac{h(t,\mu,\nu)^{\delta/2}}{(\frac{1}{\mu}\gamma_1-\beta_1)^{\delta/2}},$$

then,

$$\|x(t)\|^{2+\delta}\leq\frac{h(t,\mu,\nu)^{1+\delta/2}}{(\frac{1}{\mu}\gamma_1-\beta_1)^{1+\delta/2}}.$$

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It follows that

$$\begin{aligned}\frac{d}{dt}h(t,\mu,\nu) \;\leq\; & -\frac{1-2\mu\beta}{\frac{1}{\mu}\|H\|+\beta_1}h(t,\mu,\nu)\\ & +\frac{2|\nu|k\left(\frac{1}{\mu}\|H\|+\beta_1\right)}{(\frac{1}{\mu}\gamma_1-\beta_1)^{1+\delta/2}}h(t,\mu,\nu)^{1+\delta/2}\\ & +2|\nu|r\frac{\left(\frac{\|H\|}{\mu}+\beta_1\right)}{\sqrt{\frac{1}{\mu}\gamma_1-\beta_1}}\sqrt{h(t,\mu,\nu)}.\end{aligned}$$

Introduce the following notation

$$\epsilon_1=\frac{1-2\mu\beta}{\frac{1}{\mu}\|H\|+\beta_1},\ \epsilon_2=\frac{2|\nu|k\left(\frac{1}{\mu}\|H\|+\beta_1\right)}{(\frac{1}{\mu}\gamma_1-\beta_1)^{1+\delta/2}}\quad and\quad \epsilon_3=2|\nu|r\frac{\left(\frac{\|H\|}{\mu}+\beta_1\right)}{\sqrt{\frac{1}{\mu}\gamma_1-\beta_1}},$$

hence

$$\frac{d}{dt}h(t,\mu,\nu)\leq-\epsilon_1 h(t,\mu,\nu)+\epsilon_2 h(t,\mu,\nu)^{1+\delta/2}+\epsilon_3\sqrt{h(t,\mu,\nu)}.$$

Let

$$z(t)=\sqrt{h(t,\mu,\nu)},$$

we have

$$\frac{d}{dt}z(t)\leq-\frac{\epsilon_1}{2}z(t)+\frac{\epsilon_2}{2}z(t)^{1+\delta}+\frac{\epsilon_3}{2}. \tag{13}$$

Let $z(t,\varepsilon)$ the solution of (13) where $\varepsilon=(\epsilon_1,\epsilon_2,\epsilon_3)\in\mathbb{R}^3_+$ and $y_1(t,\varepsilon_0)$ the solution of

$$\frac{d}{dt}z(t)\leq-\frac{\epsilon_1}{2}z(t)+\frac{\epsilon_2}{2}z(t)^{1+\delta} \tag{14}$$

where $\varepsilon_0=(\epsilon_1,\epsilon_2,0)\in\mathbb{R}^3_+$.

In order to solve (14), we can take $\eta=1+\delta$ and $w(t)=y_1(t,\varepsilon_0)^{1-\eta}=y_1(t,\varepsilon_0)^{-\delta}$. Thus,

$$\frac{d}{dt}w(t)=\frac{\epsilon_1\delta}{2}w-\frac{\epsilon_2\delta}{2}.$$

Solving the homogenous equation

$$\frac{d}{dt}w(t)=\frac{\epsilon_1\delta}{2}w,$$

we get

$$w(t)=L\ e^{\frac{\epsilon_1\delta}{2}t}.$$

Now, suppose that L is a function that depends on t, i.e. we have

$$w(t)=L(t)\ e^{\frac{\epsilon_1\delta}{2}t}.$$

A simple computation shows that

$$L(t) = \frac{\epsilon_2}{\epsilon_1}\, e^{-\frac{\epsilon_1\delta}{2}t} + \theta, \quad \forall\theta \geq 0,$$

and consequently

$$w(t) = \frac{\epsilon_2}{\epsilon_1} + \theta e^{\frac{\epsilon_1\delta}{2}t}$$

where

$$\theta = \left(w(t_0) - \frac{\epsilon_2}{\epsilon_1}\right)\, e^{-\frac{\epsilon_1\delta}{2}t_0}.$$

It follows that,

$$w(t) = \frac{\epsilon_2}{\epsilon_1} + \left(w(t_0) - \frac{\epsilon_2}{\epsilon_1}\right)\, e^{\frac{\epsilon_1\delta}{2}(t-t_0)}.$$

Since $y_1(t,\varepsilon_0) = w(t)^{-1/\delta}$ and $w(t_0) = y_1(t_0,\varepsilon_0)^{-\delta}$, we obtain

$$y_1(t,\varepsilon_0) = \left(y_1(t_0,\varepsilon_0)^{-\delta}\, e^{\frac{\epsilon_1\delta}{2}(t-t_0)} + \frac{\epsilon_2}{\epsilon_1} - \frac{\epsilon_2}{\epsilon_1}\, e^{\frac{\epsilon_1\delta}{2}(t-t_0)}\right)^{-1/\delta}.$$

If

$$\epsilon_2 y_1^{\delta}(t_0,\varepsilon_0) < \epsilon_1, \tag{15}$$

which will be verified later on, and using the fact that for all $a \geq 0$ and $b \geq 0$, we have

$$(a+b)^p \leq a^p(1+\frac{b}{a})^p, \quad \forall p \in \mathbb{R},$$

Thus,

$$y_1(t,\varepsilon_0) \leq y_1(t_0,\varepsilon_0)e^{-\frac{\epsilon_1}{2}(t-t_0)} \quad\times\quad \left(1 - y_1^{\delta}(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1} + y_1^{\delta}(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1}e^{-\frac{\epsilon_1}{2}\delta(t-t_0)}\right)^{-1/\delta}$$

yields,

$$y_1(t,\varepsilon_0) \leq y_1(t_0,\varepsilon_0)e^{-\frac{\epsilon_1}{2}(t-t_0)}\left(1 - y_1^{\delta}(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1}\right)^{-1/\delta}.$$

Then, by the Lemma, for $\|\epsilon_3\|_2 < \gamma$ and $\lambda > 0$, we get

$$\|z(t,\varepsilon) - y_1(t,\varepsilon_0)\| < \lambda,$$

which implies that

$$\begin{aligned}
\|z(t,\varepsilon)\| &< \lambda + \left\| y_1(t_0,\varepsilon_0)e^{-\frac{\epsilon_1}{2}(t-t_0)}\left(1 - y_1^{\delta}(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1}\right)^{-1/\delta}\right\| \\
&< \lambda + (\|z(t_0,\varepsilon)\| + \gamma)e^{-\frac{\epsilon_1}{2}(t-t_0)}\left\|1 - y_1^{\delta}(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1}\right\|^{-1/\delta} \\
&< \lambda + (\sqrt{\frac{\|H\|}{\mu}}\|x(t_0)\| + \gamma)e^{-\frac{\epsilon_1}{2}(t-t_0)}\left\|1 - y_1^{\delta}(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1}\right\|^{-1/\delta}.
\end{aligned}$$

Since

$$\sqrt{\frac{\gamma_1}{\mu}-\beta_1}\|x(t)\| \leq z(t,\varepsilon) \leq \sqrt{\frac{\|H\|}{\mu}+\beta_1}\|x(t)\|,$$

then,

$$\begin{aligned} \|x(t)\| \;&\leq\; \sqrt{\frac{\|H\|}{\gamma_1-\mu\beta_1}}\left\|1-y_1^{\delta}(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1}\right\|^{-1/\delta}\|x(t_0)\|e^{-\frac{\epsilon_1}{2}(t-t_0)} \\ &\quad+\frac{\lambda}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}}+\frac{\gamma}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}}\left\|1-y_1^{\delta}(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1}\right\|^{-1/\delta}. \end{aligned}$$

The last inequality implies that the solutions of system (6) converge exponentially toward the ball $B(0,r_2)$ whose radius is given by

$$r_2=\frac{\lambda}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}}+\frac{\gamma}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}}\left\|1-y_1^{\delta}(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1}\right\|^{-1/\delta}$$

which is clearly positive.

Finally, let verify the condition (15). Since $|\nu|<\nu_0$, $0<\mu<\mu_0$ and $\|x(t_0)\|\leq\rho$, then

$$\begin{aligned} \frac{\epsilon_2}{\epsilon_1}y_1^{\delta}(t_0,\varepsilon_0) \;&\leq\; \frac{2|\nu|k\left(\frac{\|H\|}{\mu}+\beta_1\right)^2}{(\frac{1}{\mu}\gamma_1-\beta_1)^{1+\delta/2}\,(1-2\mu\beta)}(\|z(t_0,\varepsilon)\|+\gamma))^{\delta} \\ &\leq\; \frac{2\nu_0 k}{\mu^{1-\delta/2}}\,\frac{(\|H\|+\mu\beta_1)^2}{(\gamma_1-\mu\beta_1)^{1+\delta/2}\,(1-2\mu\beta)}\left(\sqrt{\frac{\|H\|}{\mu}}\rho+\gamma\right)^{\delta}. \end{aligned}$$

Hence, according to the definition of ν_0, we have

$$\frac{\epsilon_2}{\epsilon_1}y_1^{\delta}(t_0,\varepsilon_0)<1.$$

4 Application to control

In this section we study the stabilization problem of a control system modeled by the same dynamic as (6).

Definition 6. *A function $\alpha:[0,a[\to[0,+\infty[$ is said to be of class $\mathcal{K}$, if it is continuous, strictly increasing and $\alpha(0)=0$. It is of class $\mathcal{K}_\infty$ if, in addition, $a=+\infty$ and $\alpha(r)\to+\infty$ as $r\to+\infty$.*

Let as recall the following result (see [6]).

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Theorem 3. *Let consider system* (1) *and suppose that there exist a continuously differentiable real function $h(\cdot,\cdot)$ on $\mathbb{R}_+ \times \mathbb{R}^n$, $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, a $\mathcal{K}$ function $\alpha_3(\cdot)$ and a small positive real number ϱ such that the following inequalities hold for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$*

$$\alpha_1(\|x\|) \le h(t,x) \le \alpha_2(\|x\|)$$

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} f(t,x) \le -\alpha_3(\|x\|) + \varrho.$$

Then the system is globally uniformly practically stable with $r = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\varrho)$.

When the function satisfying $f(t,0) \neq 0$ for certain $t \in \mathbb{R}_+$, we shall study the asymptotic stability of the system at a neighborhood of the origin viewed as a small ball centered at the origin. The state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner. The following result gives sufficient conditions for practical global exponential stability.

Theorem 4. *Consider system* (1)*. Let $h : [0,+\infty[\times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable Lyapunov function such that*

$$c_1\|x\|^2 \le h(t,x) \le c_2\|x\|^2$$

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} f(t,x) \le -c_3 h(t,x) + \varrho$$

for all $t \ge 0$ and $x \in \mathbb{R}^n$, where c_1, c_2 and c_3 are positive constants. Then B_r is globally uniformly exponentially stable, with $r = \sqrt{\varrho/c_1c_2}$.

Now we state the stabilizability problem associated with the following nonlinear time-varying control system:

$$\frac{dx}{dt} = f(t,x(t),u(t)), \quad t \ge 0, \tag{16}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f(t,x,u) : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$.

Definition 7. *The feedback controller $u(t) = u(t,x(t))$, where $u(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^m$ stabilizes globally uniformly asymptotically or exponentially the control system* (16) *if the closed-loop system*

$$\frac{dx}{dt} = f(t,x(t),u(t,x(t))) \tag{17}$$

is globally uniformly asymptotic or exponential stable.

In the case where $f(t,0,0) \neq 0$ for a certain $t \ge 0$. We can formulate the above definition as:

Definition 8. *The feedback controller $u(t) = u(t,x(t))$ stabilizes globally uniformly asymptotically or exponentially the control system* (16) *with respect B_r, if the associated closed-loop system* (17) *is globally practically uniformly asymptotically or exponentially stable.*

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From Theorem 3, one has the following result which concern the asymptotic stabilizability problem of system (16).

Theorem 5. *Suppose that there exist a stabilizing feedback controller* $u(t) = u(t, x(t))$ *for control system* (16) *and a continuously differentiable function* $h(\cdot, \cdot) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, $\mathcal{K}_\infty$ *functions* $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, *a* $\mathcal{K}$ *function* $\alpha_3(\cdot)$ *and a small positive real number* ϱ *such that the following inequalities hold for all* $t \in \mathbb{R}_+$ *and* $x \in \mathbb{R}^n$

$$\alpha_1(\|x\|) \leq h(t, x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial h}{\partial x}\, f(t, x, u(t, x(t))) \leq -\alpha_3(\|x\|) + \varrho.$$

Then system (16) *in closed-loop with the feedback controller* $u = u(t, x(t))$ *is globally uniformly practically asymptotically stable with* $r = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\varrho)$.

Also, we can say that the control system (16) is globally uniformly exponentially stabilizable by the feedback control $u(t) = u(t, x(t))$, where $u(t, x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^m$, if the closed-loop system (17) is globally uniformly exponentially stable.

Definition 9. B_r *is globally uniformly exponentially stabilizable by the feedback control* $u(t) = u(t, x(t))$ *if there exist* $\gamma > 0$ *and* $k > 0$ *such that for all* $t \geq t_0 \geq 0$ *and* $x_0 \in \mathbb{R}^n$, *the solution* $x(t)$ *of the closed-loop system* (17) *satisfies:*

$$\|x(t)\| \leq k\|x_0\| exp(-\gamma(t - t_0)) + r.$$

In this case, system (16) *is globally practically uniformly exponentially stabilizable by the feedback control* $u(t) = u(t, x(t))$.

One has the following result which concern the exponential stabilizability problem of system (16).

Theorem 6. *Let* $u = u(t, x(t))$ *an exponential stabilizing feedback law and*

$$h : [0, +\infty[\times \mathbb{R}^n \to \mathbb{R}$$

be a continuously differentiable Lyapunov function such that

$$c_1\|x\|^2 \leq h(t, x) \leq c_2\|x\|^2$$

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\, f(t, x, u(t, x(t))) \leq -c_3 h(t, x) + \varrho$$

for all $t \geq 0$ *and* $x \in \mathbb{R}^n$, *where* c_1, c_2 *and* c_3 *are positive constants. Then* B_r *is globally uniformly exponentially stable with* $r = \sqrt{\varrho / c_1 c_2}$, *with respect the closed-loop system* (17).

Now, we will study the practical exponential stability problem a class of nonlinear systems of the form (6). It is worth to notice that the origin is not required to be an equilibrium point for the system (6). This may be in many situations meaningful from

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a practical point of view specially, when stability for control systems is investigated.

Consider the class of systems that can be modeled by:

$$\frac{dx}{dt} = \mu(A(\alpha(t)) + B(t))x + \nu\varphi(t, x, u), \quad t \geq 0, \tag{18}$$

where $A(\alpha(t)) \in \mathbb{R}^{n\times n}$ is a matrix given by $A(\alpha(t)) = \alpha_1(t)A_1 + \alpha_2(t)A_2$, with $\alpha_1(t)+\alpha_2(t) = 1, \alpha_i(t) \in \mathbb{R}^+,\ \forall t \geq 0,\ B(t) \in \mathbb{R}^{n\times n}$ is T-periodic matrix, $\mu \in \mathbb{R}$, $\nu \in \mathbb{R}$ are parameters and $\varphi(t, x, u)$ is a smooth vector function. u denotes the control of the system. We suppose that there exists a stabilizing feedback control $u(t) = u(t, x(t))$, where the function u is a suitable feedback controller such that the condition (7) is replaced as follows: $\varphi(t, x, u)$ is a smooth vector function such that, for all $t \geq 0$ and $x \in \mathbb{R}^n$,

$$\varphi(t + T, x, u(t, x(t))) = \varphi(t, x, u(t, x(t)))$$

and

$$\|\varphi(t, x, u(t, x(t)))\| \leq k\|x\|^{1+\delta} + r, \quad \delta \geq 0,\ k > 0,\ r > 0.$$

The practical uniform exponential stability can therefore be established as in Theorem 2, an d an estimation as in Definition 9 can be obtained which gives that the system (18) in closed-loop with $u(t) = u(t, x(t))$ is practically globally uniformly exponentially stable.

5 Conclusion

Asymptotic stability of a class of parametric differential equations has been studied. New sufficient conditions for practical uniform asymptotic exponential stability of solutions for parametric systems with periodic coefficients are obtained. An application to control system is given.

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Generalized Canavati Fractional Hilbert-Pachpatte type inequalities for Banach algebra valued functions

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Abstract

Using generalized Canavati fractional left and right vectorial Taylor formulae we prove corresponding left and right fractional Hilbert-Pachpatte type inequalities for Banach algebra valued functions. We cover also the sequential fractional case. We finish with applications.

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1 Introduction

Motivation follows:

We need

Definition 1 *(see [5]) A definition of the Hausdorff measure h_α goes as follows: if (T,d) is a metric space, $A \subseteq T$ and $\delta > 0$, let $\Lambda(A,\delta)$ be the set of all arbitrary collections $(C)_i$ of subsets of T, such that $A \subseteq \cup_i C_i$ and $diam(C_i) \leq \delta$ (diam =diameter) for every i. Now, for every $\alpha > 0$ define*

$$h_\alpha^\delta(A) := \inf\left\{\sum (diamC_i)^\alpha \mid (C_i)_i \in \Lambda(A,\delta)\right\}. \tag{1}$$

Then there exists $\lim\limits_{\delta\to 0} h_\alpha^\delta(A) = \sup\limits_{\delta>0} h_\alpha^\delta(A)$, and $h_\alpha(A) := \lim\limits_{\delta\to 0} h_\alpha^\delta(A)$ gives an outer measure on the power set $\mathcal{P}(T)$, which is countably additive on the σ-field

of all Borel subsets of T. If $T=\mathbb{R}^n$, then the Hausdorff measure h_n, restricted to the σ-field of the Borel subsets of $\mathbb{R}^n$, equals the Lebesgue measure on $\mathbb{R}^n$ up to a constant multiple. In particular, $h_1(C)=\mu(C)$ for every Borel set $C\subseteq\mathbb{R}$, where μ is the Lebesgue measure.

We also need

Definition 2 *([2], Ch. 1) Let $[a,b]\subset\mathbb{R}$, X be a Banach space, $\nu>0$; $n:=\lceil\nu\rceil\in\mathbb{N}$, $\lceil\cdot\rceil$ is the ceiling of the number, $f:[a,b]\to X$. We assume that $f^{(n)}\in L_1([a,b],X)$. We call the Caputo-Bochner left fractional derivative of order ν:*

$$\left(D_{*a}^{\nu}f\right)(x):=\frac{1}{\Gamma(n-\nu)}\int_a^x(x-t)^{n-\nu-1}f^{(n)}(t)\,dt,\quad\forall\ x\in[a,b].\tag{2}$$

If $\nu\in\mathbb{N}$, we set $D_{*a}^{\nu}f:=f^{(\nu)}$ the ordinary X-valued derivative, and also set $D_{*a}^{0}f:=f$. Here Γ is the gamma function and integrals are of Bochner type [3].

By [2], Ch. 1, $\left(D_{*a}^{\nu}f\right)(x)$ exists almost everywhere in $x\in[a,b]$ and $D_{*a}^{\nu}f\in L_1([a,b],X)$.

If $\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}<\infty$, then by [2], Ch. 1, $D_{*a}^{\nu}f\in C([a,b],X)$.

We are motivated by a Hilbert-Pachpatte left fractional inequality:

Theorem 3 *([2], Ch. 1) Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, and $\nu_1>\frac{1}{q}$, $\nu_2>\frac{1}{p}$, $n_i:=\lceil\nu_i\rceil$, $i=1,2$. Here $[a_i,b_i]\subset\mathbb{R}$, $i=1,2$; X is a Banach space. Let $f_i\in C^{n_i-1}([a_i,b_i],X)$, $i=1,2$. Set*

$$F_{x_i}(t_i):=\sum_{j_i=0}^{n_i-1}\frac{(x_i-t_i)^{j_i}}{j_i!}f_i^{(j_i)}(t_i),\tag{3}$$

$\forall\ t_i\in[a_i,x_i]$, where $x_i\in[a_i,b_i]$; $i=1,2$. Assume that $f_i^{(n_i)}$ exists outside a μ-null Borel set $B_{x_i}\subseteq[a_i,x_i]$, such that

$$h_1\left(F_{x_i}\left(B_{x_i}\right)\right)=0,\ \ \forall\ x_i\in[a_i,b_i];\ i=1,2.\tag{4}$$

We also assume that $f_i^{(n_i)}\in L_1([a_i,b_i],X)$, and

$$f_i^{(k_i)}(a_i)=0,\ \ k_i=0,1,...,n_i-1;\ i=1,2,\tag{5}$$

and

$$\left(D_{*a_1}^{\nu_1}f_1\right)\in L_q([a_1,b_1],X),\quad\left(D_{*a_2}^{\nu_2}f_2\right)\in L_p([a_2,b_2],X).\tag{6}$$

Then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\|f_1(x_1)\|\,\|f_2(x_2)\|\,dx_1dx_2}{\left(\frac{(x_1-a_1)^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(x_2-a_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)}\leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\nu_1)\Gamma(\nu_2)}\left\|D_{*a_1}^{\nu_1}f_1\right\|_{L_q([a_1,b_1],X)}\left\|D_{*a_2}^{\nu_2}f_2\right\|_{L_p([a_2,b_2],X)}.\tag{7}$$

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We need

Definition 4 *([2], Ch. 2) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a,b], X)$, where $f : [a,b] \to X$. We call the Caputo-Bochner right fractional derivative of order α:*

$$\left(D_{b-}^{\alpha} f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (J-x)^{m-\alpha-1} f^{(m)}(J)\, dJ, \quad \forall\, x \in [a,b]. \tag{8}$$

We observe that $D_{b-}^{m} f(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $D_{b-}^{0} f(x) = f(x)$.

By [2], Ch. 2, $\left(D_{b-}^{\alpha} f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha} f\right) \in L_1([a,b], X)$.

If $\left\|f^{(m)}\right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, then by [2], Ch. 2, $D_{b-}^{\alpha} f \in C([a,b], X)$, hence $\left\|D_{b-}^{\alpha} f\right\| \in C([a,b])$.

We are motivated also by the following Hilbert-Pachpatte right fractional inequality:

Theorem 5 *([2], Ch. 2) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\alpha_1 > \frac{1}{q}$, $\alpha_2 > \frac{1}{p}$, $m_i := \lceil \alpha_i \rceil$, $i = 1,2$. Here $[a_i, b_i] \subset \mathbb{R}$, $i = 1,2$; X is a Banach space. Let $f_i \in C^{m_i - 1}([a_i, b_i], X)$, $i = 1,2$. Set*

$$F_{x_i}(t_i) := \sum_{j_i=0}^{m_i-1} \frac{(x_i - t_i)^{j_i}}{j_i!} f_i^{(j_i)}(t_i), \tag{9}$$

$\forall\, t_i \in [x_i, b_i]$, where $x_i \in [a_i, b_i]$; $i = 1,2$. Assume that $f_i^{(m_i)}$ exists outside a μ-null Borel set $B_{x_i} \subseteq [x_i, b_i]$, such that

$$h_1\left(F_{x_i}\left(B_{x_i}\right)\right) = 0, \quad \forall\, x_i \in [a_i, b_i];\ i = 1,2. \tag{10}$$

We also assume that $f_i^{(m_i)} \in L_1([a_i, b_i], X)$, and

$$f_i^{(k_i)}(b_i) = 0, \quad k_i = 0, 1, ..., , m_i - 1;\ i = 1,2, \tag{11}$$

and

$$\left(D_{b_1-}^{a_1} f_1\right) \in L_q([a_1, b_1], X), \quad \left(D_{b_2-}^{\alpha_2} f_2\right) \in L_p([a_2, b_2], X). \tag{12}$$

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\|f_1(x_1)\| \|f_2(x_2)\|\, dx_1 dx_2}{\left(\frac{(b_1-x_1)^{p(\alpha_1-1)+1}}{p(p(\alpha_1-1)+1)} + \frac{(b_2-x_2)^{q(\alpha_2-1)+1}}{q(q(\alpha_2-1)+1)}\right)} \leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \left\|D_{b_1-}^{\alpha_1} f_1\right\|_{L_q([a_1,b_1],X)} \left\|D_{b_2-}^{\alpha_2} f_2\right\|_{L_p([a_2,b_2],X)}. \tag{13}$$

In this work we derive Hilbert-Pachpatte inequalities for Banach algebra valued functions with respect to their Canavati type generalized left and right fractional derivatives. We cover also the sequential fractional case. We finish with applications.

3

2 Background on Vectorial generalized Canavati fractional calculus

All in this section come from [2], pp. 109-115 and [1].

Let $g : [a, b] \to \mathbb{R}$ be a strictly increasing function. such that $g \in C^1([a, b])$, and $g^{-1} \in C^n([g(a), g(b)])$, $n \in \mathbb{N}$, $(X, \|\cdot\|)$ is a Banach space. Let $f \in C^n([a, b], X)$, and call $l := f \circ g^{-1} : [g(a), g(b)] \to X$. It is clear that $l, l', ..., l^{(n)}$ are continuous functions from $[g(a), g(b)]$ into $f([a, b]) \subseteq X$.

Let $\nu \geq 1$ such that $[\nu] = n$, $n \in \mathbb{N}$ as above, where $[\cdot]$ is the integral part of the number.

Clearly when $0 < \nu < 1$, $[\nu] = 0$.

I) Let $h \in C([g(a), g(b)], X)$, we define the left Riemann-Liouville Bochner fractional integral as

$$(J_\nu^{z_0} h)(z) := \frac{1}{\Gamma(\nu)} \int_{z_0}^{z} (z - t)^{\nu - 1} h(t)\, dt, \tag{14}$$

for $g(a) \leq z_0 \leq z \leq g(b)$, where Γ is the gamma function; $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu - 1} dt$. We set $J_0^{z_0} h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C_{g(x_0)}^{\nu}([g(a), g(b)], X)$ of $C^{[\nu]}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ as:

$$C_{g(x_0)}^{\nu}([g(a), g(b)], X) =$$

$$\left\{ h \in C^{[\nu]}([g(a), g(b)], X) : J_{1-\alpha}^{g(x_0)} h^{([\nu])} \in C^1([g(x_0), g(b)], X) \right\}. \tag{15}$$

So let $h \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, we define the left g-generalized X-valued fractional derivative of h of order ν, of Canavati type, over $[g(x_0), g(b)]$ as

$$D_{g(x_0)}^{\nu} h := \left(J_{1-\alpha}^{g(x_0)} h^{([\nu])} \right)'. \tag{16}$$

Clearly, for $h \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, there exists

$$\left(D_{g(x_0)}^{\nu} h \right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z - t)^{-\alpha} h^{([\nu])}(t)\, dt, \tag{17}$$

for all $g(x_0) \leq z \leq g(b)$.

In particular, when $f \circ g^{-1} \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, we have that

$$\left(D_{g(x_0)}^{\nu} \left(f \circ g^{-1} \right) \right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z - t)^{-\alpha} \left(f \circ g^{-1} \right)^{([\nu])}(t)\, dt, \tag{18}$$

for all $g(x_0) \leq z \leq g(b)$. We have that $D_{g(x_0)}^{n} \left(f \circ g^{-1} \right) = \left(f \circ g^{-1} \right)^{(n)}$ and $D_{g(x_0)}^{0} \left(f \circ g^{-1} \right) = f \circ g^{-1}$, see [1].

By [1], we have for $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ the following left generalized g-fractional, of Canavati type,X-valued Taylor's formula:

Theorem 6 *Let $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ is fixed.*

(i) If $\nu \geq 1$, then

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x_0))}{k!} (g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{19}$$

for all $x_0 \leq x \leq b$.

(ii) If $0 < \nu < 1$, we get

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{20}$$

for all $x_0 \leq x \leq b$.

II) Let $h \in C([g(a), g(b)], X)$, we define the right Riemann-Liouville Bochner fractional integral as

$$\left(J^{\nu}_{z_0-} h\right)(z) := \frac{1}{\Gamma(\nu)} \int_{z}^{z_0} (t - z)^{\nu-1} h(t)\, dt, \tag{21}$$

for $g(a) \leq z \leq z_0 \leq g(b)$. We set $J^0_{z_0-} h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$ of $C^{[\nu]}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ as:

$$C^{\nu}_{g(x_0)-}([g(a), g(b)], X) :=$$

$$\left\{h \in C^{[\nu]}([g(a), g(b)], X) : J^{1-\alpha}_{g(x_0)-} h^{([\nu])} \in C^1([g(a), g(x_0)], X)\right\}. \tag{22}$$

So let $h \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, we define the right g-generalized X-valued fractional derivative of h of order ν, of Canavati type, over $[g(a), g(x_0)]$ as

$$D^{\nu}_{g(x_0)-} h := (-1)^{n-1} \left(J^{1-\alpha}_{g(x_0)-} h^{([\nu])}\right)'. \tag{23}$$

Clearly, for $h \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, there exists

$$\left(D^{\nu}_{g(x_0)-} h\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{z}^{g(x_0)} (t - z)^{-\alpha} h^{([\nu])}(t)\, dt, \tag{24}$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

In particular, when $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, we have that

$$\left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_z^{g(x_0)} (t-z)^{-\alpha} \left(f \circ g^{-1}\right)^{([\nu])}(t)\, dt, \tag{25}$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

We get that

$$\left(D^{n}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = (-1)^n \left(f \circ g^{-1}\right)^{(n)}(z) \tag{26}$$

and $\left(D^{0}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = \left(f \circ g^{-1}\right)(z)$, all $z \in [g(a), g(b)]$, see [1].

By [1], we have for $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ is fixed, the following right generalized g-fractional, of Canavati type, X-valued Taylor's formula:

Theorem 7 *Let $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ is fixed.*

(i) If $\nu \geq 1$, then

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x_0))}{k!} (g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{27}$$

for all $a \leq x \leq x_0$,

(ii) If $0 < \nu < 1$, we get

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{28}$$

all $a \leq x \leq x_0$.

III) Denote by

$$D^{m\nu}_{g(x_0)} = D^{\nu}_{g(x_0)} D^{\nu}_{g(x_0)} \ldots D^{\nu}_{g(x_0)} \quad (m\text{-times}),\ m \in \mathbb{N}. \tag{29}$$

We mention the following modified and generalized left X-valued fractional Taylor's formula of Canavati type:

Theorem 8 *Let $f \in C^1([a, b], X)$, $g \in C^1([a, b])$, strictly increasing: $g^{-1} \in C^1([g(a), g(b)])$. Assume that $\left(D^{i\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right) \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, $0 < \nu < 1$, $x_0 \in [a, b]$, for $i = 0, 1, \ldots, m$. Then*

$$f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x_0)}^{g(x)} (g(x) - z)^{(m+1)\nu-1} \left(D^{(m+1)\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(z)\, dz, \tag{30}$$

all $x_0 \leq x \leq b$.

IV) Denote by

$$D_{g(x_0)-}^{m\nu} = D_{g(x_0)-}^{\nu} D_{g(x_0)-}^{\nu} ... D_{g(x_0)-}^{\nu} \quad (m \text{ times}),\ m \in \mathbb{N}. \tag{31}$$

We mention the following modified and generalized right X-valued fractional Taylor's formula of Canavati type:

Theorem 9 *Let $f \in C^1([a,b], X)$, $g \in C^1([a,b])$, strictly increasing: $g^{-1} \in C^1([g(a), g(b)])$. Assume that $\left(D_{g(x_0)-}^{i\nu}\left(f \circ g^{-1}\right)\right) \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, $0 < \nu < 1$, $x_0 \in [a,b]$, for all $i = 0, 1, ..., m$. Then*

$$f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x)}^{g(x_0)} (z - g(x))^{(m+1)\nu - 1} \left(D_{g(x_0)-}^{(m+1)\nu}\left(f \circ g^{-1}\right)\right)(z)\, dz, \tag{32}$$

all $a \leq x \leq x_0 \leq b$.

3 Banach Algebras background

All here come from [4].

We need

Definition 10 *([4], p. 245) A complex algebra is a vector space A over the complex filed $\mathbb{C}$ in which a multiplication is defined that satisfies*

$$x(yz) = (xy)z, \tag{33}$$

$$(x+y)z = xz + yz, \quad x(y+z) = xy + xz, \tag{34}$$

and

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \tag{35}$$

for all x, y and z in A and for all scalars α.

Additionally if A is a Banach space with respect to a norm that satisfies the multiplicative inequality

$$\|xy\| \leq \|x\| \|y\| \quad (x \in A,\ y \in A) \tag{36}$$

and if A contains a unit element e such that

$$xe = ex = x \quad (x \in A) \tag{37}$$

and

$$\|e\| = 1, \tag{38}$$

then A is called a Banach algebra.

A is commutative iff $xy = yx$ for all $x, y \in A$.

We make

Remark 11 *Commutativity of A will be explicited stated when needed.*

There exists at most one $e \in A$ that satisfies (37).

Inequality (36) makes multiplication to be continuous, more precisely left and right continuous, see [4], p. 246.

Multiplication in A is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.

For nice examples about Banach algebras see [4], p. 247-248, § 10.3.

We also make

Remark 12 *Next we mention about integration of A-valued functions, see [4], p. 259, § 10.22:*

If A is a Banach algebra and f is a continuous A-valued function on some compact Hausdorff space Q on which a complex Borel measure μ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [4], simply because A is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then

$$x \int_Q f \ d\mu = \int_Q x f(p) \ d\mu(p) \tag{39}$$

and

$$\left(\int_Q f \ d\mu \right) x = \int_Q f(p) x \ d\mu(p). \tag{40}$$

The Bochner integrals we will involve in our article follow (39) and (40). Also, let $f \in C([a,b], X)$, where $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ is a Banach space. By [2], p. 3, f is Bochner integrable.

4 Main Results

We start with a left generalized Canavati fractional Hilbert-Pachpatte type inequality over a Banach algebra.

Theorem 13 *Let $p, q > 1$, such that $\frac{1}{p} + \frac{1}{q} = 1$, and $(A, \|\cdot\|)$ is a Banach algebra; and $i = 1, 2$. Let also $x_{0i} \in [a_i, b_i] \subset \mathbb{R}$, $\nu_i \geq 1$, $n_i = [\nu_i]$, $f_i \in C^{n_i}([a_i, b_i], A)$; $g_i \in C^1([a_i, b_i])$, strictly increasing, such that $g_i^{-1} \in C^{n_i}([g_i(a_i), g_i(b_i)])$, with $\left(f_i \circ g_i^{-1}\right)^{(k_i)}(g_i(x_{0i})) = 0$, $k_i = 0, 1, ..., n_i - 1$. Assume further that $f_i \circ g_i^{-1} \in C^{\nu_i}_{g_i(x_{0i})}([g_i(a_i), g_i(b_i)], A)$. Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)} \int_{g_2(x_{02})}^{g_2(b_2)} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(z_1 - g_1(x_{01}))^{p(\nu_1 - 1) + 1}}{p(p(\nu_1 - 1) + 1)} + \frac{(z_2 - g_2(x_{02}))^{q(\nu_2 - 1) + 1}}{q(q(\nu_2 - 1) + 1)} \right)} \leq$$

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$$\frac{\left(g_1\left(b_1\right)-g_1\left(x_{01}\right)\right)\left(g_2\left(b_2\right)-g_2\left(x_{02}\right)\right)}{\Gamma\left(\nu_1\right)\Gamma\left(\nu_2\right)} \tag{41}$$

$$\left\|\left\|D^{\nu_1}_{g_1(x_{01})}\left(f_1\circ g_1^{-1}\right)\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)],A)}\left\|\left\|D^{\nu_2}_{g_2(x_{02})}\left(f_2\circ g_2^{-1}\right)\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)],A)}.$$

Proof. By (19) and assumptions we get that

$$\left(f_i\circ g_i^{-1}\right)\left(z_i\right)=\frac{1}{\Gamma\left(\nu_i\right)}\int_{g_i(x_{0i})}^{z_i}\left(z_i-t_i\right)^{\nu_i-1}\left(D^{\nu_i}_{g_i(x_{0i})}\left(f_i\circ g_i^{-1}\right)\right)\left(t_i\right)dt_i, \tag{42}$$

for all $g_i\left(x_{0i}\right)\leq z_i\leq g_i\left(b_i\right)$; $i=1,2$.

By Hölder's inequality we obtain

$$\left\|\left(f_1\circ g_1^{-1}\right)\left(z_1\right)\right\|\leq\frac{1}{\Gamma\left(\nu_1\right)}\int_{g_1(x_{01})}^{z_1}\left(z_1-t_1\right)^{\nu_1-1}\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1\circ g_1^{-1}\right)\right)\left(t_1\right)\right\|dt_1\leq$$

$$\frac{1}{\Gamma\left(\nu_1\right)}\left(\int_{g_1(x_{01})}^{z_1}\left(z_1-t_1\right)^{p(\nu_1-1)}dt_1\right)^{\frac{1}{p}}\left(\int_{g_1(x_{01})}^{z_1}\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1\circ g_1^{-1}\right)\right)\left(t_1\right)\right\|^q dt_1\right)^{\frac{1}{q}}=$$

$$\frac{1}{\Gamma\left(\nu_1\right)}\frac{\left(z_1-g_1\left(x_{01}\right)\right)^{\frac{p(\nu_1-1)+1}{p}}}{\left(p\left(\nu_1-1\right)+1\right)^{\frac{1}{p}}}\left(\int_{g_1(x_{01})}^{z_1}\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1\circ g_1^{-1}\right)\right)\left(t_1\right)\right\|^q dt_1\right)^{\frac{1}{q}}. \tag{43}$$

That is

$$\left\|\left(f_1\circ g_1^{-1}\right)\left(z_1\right)\right\|\leq\frac{1}{\Gamma\left(\nu_1\right)}\frac{\left(z_1-g_1\left(x_{01}\right)\right)^{\frac{p(\nu_1-1)+1}{p}}}{\left(p\left(\nu_1-1\right)+1\right)^{\frac{1}{p}}}$$

$$\left(\int_{g_1(x_{01})}^{z_1}\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1\circ g_1^{-1}\right)\right)\left(t_1\right)\right\|^q dt_1\right)^{\frac{1}{q}}, \tag{44}$$

for all $g_1\left(x_{01}\right)\leq z_1\leq g_1\left(b_1\right)$.

Similarly, we prove that

$$\left\|\left(f_2\circ g_2^{-1}\right)\left(z_2\right)\right\|\leq\frac{1}{\Gamma\left(\nu_2\right)}\frac{\left(z_2-g_2\left(x_{02}\right)\right)^{\frac{q(\nu_2-1)+1}{q}}}{\left(q\left(\nu_2-1\right)+1\right)^{\frac{1}{q}}}$$

$$\left(\int_{g_2(x_{02})}^{z_2}\left\|\left(D^{\nu_2}_{g_2(x_{02})}\left(f_2\circ g_2^{-1}\right)\right)\left(t_2\right)\right\|^p dt_2\right)^{\frac{1}{p}}, \tag{45}$$

for all $g_2\left(x_{02}\right)\leq z_2\leq g_2\left(b_2\right)$.

Therefore we have

$$\left\|\left(f_1\circ g_1^{-1}\right)\left(z_1\right)\right\|\leq\frac{1}{\Gamma\left(\nu_1\right)}\frac{\left(z_1-g_1\left(x_{01}\right)\right)^{\frac{p(\nu_1-1)+1}{p}}}{\left(p\left(\nu_1-1\right)+1\right)^{\frac{1}{p}}}$$

$$\left\|\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1\circ g_1^{-1}\right)\right)\right\|\right\|_{q,[g_1(x_{01}),g_1(b_1)]}, \tag{46}$$

for all $g_1(x_{01})\leq z_1\leq g_1(b_1)$;

and

$$\left\|\left(f_2\circ g_2^{-1}\right)(z_2)\right\|\leq\frac{1}{\Gamma(\nu_2)}\frac{(z_2-g_2(x_{02}))^{\frac{q(\nu_2-1)+1}{q}}}{(q(\nu_2-1)+1)^{\frac{1}{q}}}$$
$$\left\|\left\|\left(D^{\nu_2}_{g_2(x_{02})}\left(f_2\circ g_2^{-1}\right)\right)\right\|\right\|_{p,[g_2(x_{02}),g_2(b_2)]}, \tag{47}$$

for all $g_2(x_{02})\leq z_2\leq g_2(b_2)$.

Hence we get that

$$\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\right\|\left\|\left(f_2\circ g_2^{-1}\right)(z_2)\right\|\leq\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)(p(\nu_1-1)+1)^{\frac{1}{p}}(q(\nu_2-1)+1)^{\frac{1}{q}}}$$
$$(z_1-g_1(x_{01}))^{\frac{p(\nu_1-1)+1}{p}}(z_2-g_2(x_{02}))^{\frac{q(\nu_2-1)+1}{q}} \tag{48}$$
$$\left\|\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1\circ g_1^{-1}\right)\right)\right\|\right\|_{q,[g_1(x_{01}),g_1(b_1)]}\left\|\left\|\left(D^{\nu_2}_{g_2(x_{02})}\left(f_2\circ g_2^{-1}\right)\right)\right\|\right\|_{p,[g_2(x_{02}),g_2(b_2)]}\leq$$

(using Young's inequality for $a,b\geq 0$, $a^{\frac{1}{p}}b^{\frac{1}{q}}\leq\frac{a}{p}+\frac{b}{q}$)

$$\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\left(\frac{(z_1-g_1(x_{01}))^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(z_2-g_2(x_{02}))^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)$$
$$\left\|\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1\circ g_1^{-1}\right)\right)\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)],A)}\left\|\left\|\left(D^{\nu_2}_{g_2(x_{02})}\left(f_2\circ g_2^{-1}\right)\right)\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)],A)},$$

$\forall\,(z_1,z_2)\in[g_1(x_{01}),g_1(b_1)]\times[g_2(x_{02}),g_2(b_2)]$.

So far we have

$$\frac{\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\left(f_2\circ g_2^{-1}\right)(z_2)\right\|}{\left(\frac{(z_1-g_1(x_{01}))^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(z_2-g_2(x_{02}))^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)}\leq \tag{50}$$

$$\frac{\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\right\|\left\|\left(f_2\circ g_2^{-1}\right)(z_2)\right\|}{\left(\frac{(z_1-g_1(x_{01}))^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(z_2-g_2(x_{02}))^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)}\leq \tag{51}$$

$$\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\left\|\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1\circ g_1^{-1}\right)\right)\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)],A)}$$
$$\left\|\left\|\left(D^{\nu_2}_{g_2(x_{02})}\left(f_2\circ g_2^{-1}\right)\right)\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)],A)},$$

$\forall\,(z_1,z_2)\in[g_1(x_{01}),g_1(b_1)]\times[g_2(x_{02}),g_2(b_2)]$.

The denominators in (50), (51) can be zero only when both $z_1=g_1(x_{01})$ and $z_2=g_2(x_{02})$.

Therefore we obtain (41), by integrating (50), (51) over $[g_1(x_{01}),g_1(b_1)]\times[g_2(x_{02}),g_2(b_2)]$. ■

We continue with a right generalized Canavati fractional Hilbert-Pachpatte type inequality over a Banach algebra.

Theorem 14 *All as in Theorem 13, however now it is* $f_i \circ g_i^{-1} \in C^{\nu_i}_{g_i(x_{0i})-}([g_i(a_i), g_i(b_i)], A)$, *for* $i = 1, 2$. *Then*

$$\int_{g_1(a_1)}^{g_1(x_{01})} \int_{g_2(a_2)}^{g_2(x_{02})} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(g_1(x_{01}) - z_1)^{p(\nu_1 - 1)+1}}{p(p(\nu_1 - 1)+1)} + \frac{(g_2(x_{02}) - z_2)^{q(\nu_2 - 1)+1}}{q(q(\nu_2 - 1)+1)} \right)} \leq$$

$$\frac{(g_1(x_{01}) - g_1(a_1))(g_2(x_{02}) - g_2(a_2))}{\Gamma(\nu_1)\Gamma(\nu_2)} \tag{52}$$

$$\left\| \left\| D^{\nu_1}_{g_1(x_{01})-} \left(f_1 \circ g_1^{-1}\right) \right\| \right\|_{L_q([g_1(a_1), g_1(x_{01})], A)} \left\| \left\| D^{\nu_2}_{g_2(x_{02})-} \left(f_2 \circ g_2^{-1}\right) \right\| \right\|_{L_p([g_2(a_2), g_2(x_{02})], A)}.$$

Proof. Similar to Theorem 13, by using now (27). ■

Next comes a sequential left generalized Canavati fractional Hilbert-Pachpatte type inequality over a Banach algebra.

Theorem 15 *Let* $p, q > 1$, *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *and* $(A, \|\cdot\|)$ *is a Banach algebra; and* $i = 1, 2$. *Let also* $f_i \in C^1([a_i, b_i], A)$; $g_i \in C^1([a_i, b_i])$, *strictly increasing, such that* $g_i^{-1} \in C^1([g_i(a_i), g_i(b_i)])$. *Assume that* $\frac{1}{(m_i+1)q} < \nu_i < 1$, $x_{0i} \in [a_i, b_i]$, *and* $D^{j_i \nu_i}_{g_i(x_{0i})} \left(f_i \circ g_i^{-1}\right) \in C^{\nu_i}_{g_i(x_{0i})}([g_i(a_i), g_i(b_i)], A)$, *for* $j_i = 0, 1, ..., m_i \in \mathbb{N}$. *Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)} \int_{g_2(x_{02})}^{g_2(b_2)} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(z_1 - g_1(x_{01}))^{p((m_1+1)\nu_1 - 1)+1}}{p(p((m_1+1)\nu_1 - 1)+1)} + \frac{(z_2 - g_2(x_{02}))^{q((m_2+1)\nu_2 - 1)+1}}{q(q((m_2+1)\nu_2 - 1)+1)} \right)} \leq$$

$$\frac{(g_1(b_1) - g_1(x_{01}))(g_2(b_2) - g_2(x_{02}))}{\Gamma((m_1+1)\nu_1)\Gamma((m_2+1)\nu_2)} \tag{53}$$

$$\left\| \left\| D^{(m_1+1)\nu_1}_{g_1(x_{01})} \left(f_1 \circ g_1^{-1}\right) \right\| \right\|_{L_q([g_1(x_{01}), g_1(b_1)], A)} \left\| \left\| D^{(m_2+1)\nu_2}_{g_2(x_{02})} \left(f_2 \circ g_2^{-1}\right) \right\| \right\|_{L_p([g_2(x_{02}), g_2(b_2)], A)}.$$

Proof. Using (30), as similar to Theorem 13 the proof is omitted. ■

The right side analog of Theorem 15 follows:

Theorem 16 *Let* $p, q > 1$, *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *and* $(A, \|\cdot\|)$ *is a Banach algebra; and* $i = 1, 2$. *Let also* $f_i \in C^1([a_i, b_i], A)$; $g_i \in C^1([a_i, b_i])$, *strictly increasing, such that* $g_i^{-1} \in C^1([g_i(a_i), g_i(b_i)])$. *Assume that* $\frac{1}{(m_i+1)q} < \nu_i < 1$, $x_{0i} \in [a_i, b_i]$, *and* $D^{j_i \nu_i}_{g_i(x_{0i})-} \left(f_i \circ g_i^{-1}\right) \in C^{\nu_i}_{g_i(x_{0i})-}([g_i(a_i), g_i(b_i)], A)$, *for* $j_i = 0, 1, ..., m_i \in \mathbb{N}$. *Then*

$$\int_{g_1(a_1)}^{g_1(x_{01})} \int_{g_2(a_2)}^{g_2(x_{02})} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(g_1(x_{01}) - z_1)^{p((m_1+1)\nu_1 - 1)+1}}{p(p((m_1+1)\nu_1 - 1)+1)} + \frac{(g_2(x_{02}) - z_2)^{q((m_2+1)\nu_2 - 1)+1}}{q(q((m_2+1)\nu_2 - 1)+1)} \right)} \leq$$

$$\frac{(g_1(x_{01}) - g_1(a_1))(g_2(x_{02}) - g_2(a_2))}{\Gamma((m_1+1)\nu_1)\Gamma((m_2+1)\nu_2)} \tag{54}$$

$$\left\| \left\| D^{(m_1+1)\nu_1}_{g_1(x_{01})-} \left(f_1 \circ g_1^{-1}\right) \right\| \right\|_{L_q([g_1(a_1), g_1(x_{01})], A)} \left\| \left\| D^{(m_2+1)\nu_2}_{g_2(x_{02})-} \left(f_2 \circ g_2^{-1}\right) \right\| \right\|_{L_p([g_2(a_2), g_2(x_{02})], A)}.$$

Proof. Using (32), as similar to Theorem 13 is omitted. ■

5 Applications

We give

Corollary 17 *(to Theorem 13) All as in Theorem 13 for $g_i(t) = e^t$, $i = 1, 2$. Then*

$$\int_{e^{x_{01}}}^{e^{b_1}} \int_{e^{x_{02}}}^{e^{b_2}} \frac{\left\| (f_1 \circ \log)(z_1)(f_2 \circ \log)(z_2) \right\| dz_1 dz_2}{\left(\frac{(z_1 - e^{x_{01}})^{p(\nu_1 - 1) + 1}}{p(p(\nu_1 - 1) + 1)} + \frac{(z_2 - e^{x_{02}})^{q(\nu_2 - 1) + 1}}{q(q(\nu_2 - 1) + 1)} \right)} \leq$$

$$\frac{\left(e^{b_1} - e^{x_{01}}\right)\left(e^{b_2} - e^{x_{02}}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)} \tag{55}$$

$$\left\| \left\| D_{e^{x_{01}}}^{\nu_1} (f_1 \circ \log) \right\| \right\|_{L_q\left(\left[e^{x_{01}}, e^{b_1}\right], A\right)} \left\| \left\| D_{e^{x_{02}}}^{\nu_2} (f_2 \circ \log) \right\| \right\|_{L_p\left(\left[e^{x_{02}}, e^{b_2}\right], A\right)}.$$

We finish with

Corollary 18 *(to Theorem 15) All as in Theorem 15 for $[a_1, b_1] \subset \mathbb{R}$, $[a_2, b_2] \subset (0, \infty)$, and $g_1(t) = e^t$ and $g_2(t) = \log t$. Then*

$$\int_{e^{x_{01}}}^{e^{b_1}} \int_{\log(x_{02})}^{\log(b_2)} \frac{\left\| (f_1 \circ \log)(z_1)\left(f_2 \circ e^t\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(z_1 - e^{x_{01}})^{p((m_1+1)\nu_1 - 1) + 1}}{p(p((m_1+1)\nu_1 - 1) + 1)} + \frac{(z_2 - \log(x_{02}))^{q((m_2+1)\nu_2 - 1) + 1}}{q(q((m_2+1)\nu_2 - 1) + 1)} \right)} \leq$$

$$\frac{\left(e^{b_1} - e^{x_{01}}\right) \log(b_2 / x_{02})}{\Gamma((m_1 + 1)\nu_1)\Gamma((m_2 + 1)\nu_2)} \tag{56}$$

$$\left\| \left\| D_{e^{x_{01}}}^{(m_1+1)\nu_1} (f_1 \circ \log) \right\| \right\|_{L_q\left(\left[e^{x_{01}}, e^{b_1}\right], A\right)} \left\| \left\| D_{\log(x_{02})}^{(m_2+1)\nu_2} \left(f_2 \circ e^t\right) \right\| \right\|_{L_p([\log(x_{02}), \log(b_2)], A)}.$$

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Generalized Ostrowski, Opial and Hilbert-Pachpatte type inequalities for Banach algebra valued functions involving integer vectorial derivatives

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Abstract

Using a generalized vectorial Taylor formula involving ordinary vector derivatives we establish mixed Ostrowski, Opial and Hilbert-Pachpatte type inequalities for several Banach algebra valued functions. The estimates are with respect to all norms $\|\cdot\|_p$, $1 \leq p \leq \infty$. We finish with applications.

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1 Introduction

The following result motivates our work.

Theorem 1 *(1938, Ostrowski [6]) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty^{\sup} := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty^{\sup}, \tag{1}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

1

Ostrowski type inequalities have great applications to integral approximations in Numerical Analysis.

We present ([1], Ch. 8,9) mixed fractional Ostrowski inequalities for several functions for various norms.

In this article we generalize [1], Ch. 8,9 for several Banach algebra valued functions by using ordinary vector valued derivatives and our integrals here are of Bochner type [4]. Motivation comes also from [3].

We are also inspired by Z. Opial [5], 1960, famous inequality.

Theorem 2 *Let $x(t) \in C^1([0,h])$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then*

$$\int_0^h |x(t)\, x'(t)|\, dt \leq \frac{h}{4} \int_0^h (x'(t))^2\, dt. \tag{2}$$

In (2), the constant $\frac{h}{4}$ is the best possible. Inequality (2) holds as equality for the optimal function

$$x(t) = \begin{cases} ct, & 0 \leq t \leq \frac{h}{2}, \\ c(h-t), & \frac{h}{2} \leq t \leq h, \end{cases} \tag{3}$$

where $c > 0$ is an arbitrary constant.

Opial-type inequalities are used a lot in proving uniqueness of solutions to differential equations and also to give upper bounds to their solutions.

In this work we also derive Opial type inequalities for Banach algebra valued functions with respect to ordinary vector valued derivatives.

Additionally we include in this article related Hilbert-Pachpatte type inequalities, [7]. We finish with selective applications to Ostrowski, Opial and Hilbert-Pachpatte inequalities.

2 About Banach Algebras

All here come from [8].

We need

Definition 3 *([8], p. 245) A complex algebra is a vector space A over the complex field $\mathbb{C}$ in which a multiplication is defined that satisfies*

$$x(yz) = (xy)z, \tag{4}$$

$$(x+y)z = xz + yz, \quad x(y+z) = xy + xz, \tag{5}$$

and

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \tag{6}$$

2

for all x, y and z in A and for all scalars α.

Additionally if A is a Banach space with respect to a norm that satisfies the multiplicative inequality

$$\|xy\| \leq \|x\| \, \|y\| \quad (x \in A, \, y \in A) \tag{7}$$

and if A contains a unit element e such that

$$xe = ex = x \quad (x \in A) \tag{8}$$

and

$$\|e\| = 1, \tag{9}$$

then A is called a Banach algebra.

A is commutative iff $xy = yx$ for all $x, y \in A$.

We make

Remark 4 *Commutativity of A will be explicited stated when needed.*

There exists at most one $e \in A$ that satisfies (8).

Inequality (7) makes multiplication to be continuous, more precisely left and right continuous, see [8], p. 246.

Multiplication in A is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.

For nice examples about Banach algebras see [8], p. 247-248, § 10.3.

We also make

Remark 5 *Next we mention about integration of A-valued functions, see [8], p. 259, § 10.22:*

If A is a Banach algebra and f is a continuous A-valued function on some compact Hausdorff space Q on which a complex Borel measure μ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [8], simply because A is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then

$$x \int_Q f \, d\mu = \int_Q x f(p) \, d\mu(p) \tag{10}$$

and

$$\left(\int_Q f \, d\mu \right) x = \int_Q f(p) \, x \, d\mu(p) . \tag{11}$$

The Bochner integrals we will involve in our article follow (10) and (11).

3

3 Background

We use the following generalized vector Taylor's formula:

Theorem 6 *([2], p. 97) Let $n \in \mathbb{N}$ and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. Let any $x, y \in [a,b]$. Then*

$$f(x) = f(y) + \sum_{i=1}^{n-1} \frac{(g(x) - g(y))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(y)) \tag{12}$$

$$+ \frac{1}{(n-1)!} \int_{g(y)}^{g(x)} (g(x) - z)^{n-1} \left(f \circ g^{-1}\right)^{(n)} (z)\, dz.$$

The derivatives here are defined similarly to the numerical ones, see [9], pp. 83-86.

The above integral is of Bochner type [4], and so are the integrals in this work. By [2], p. 3, if $f \in C([a,b], X)$ then f is Bochner integrable.

4 Main Results

We start with mixed generalized Ostrowski type inequalities for several functions that are Banach algebra valued. A uniform estimate follows.

Theorem 7 *Let $n \in \mathbb{N}$ and $f_i \in C^n([a,b], A)$, $i = 1, ..., r \in \mathbb{N} - \{1\}$; where $[a,b] \subset \mathbb{R}$ and $(A, \|\cdot\|)$ is a Banach algebra. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. We assume that $\left(f_i \circ g^{-1}\right)^{(j)} (g(x_0)) = 0$, $j = 1, ..., n-1$; $i = 1, ..., r$; where $x_0 \in [a,b]$ be fixed. Denote by*

$$E(f_1, ..., f_r)(x_0) :=$$

$$\sum_{i=1}^{r} \left[\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x)\, dx - \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) dx \right) f_i(x_0) \right]. \tag{13}$$

Then

1)

$$E(f_1, ..., f_r)(x_0) = \frac{1}{(n-1)!}$$

$$\sum_{i=1}^{r} \left[(-1)^n \left[\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{n-1} \left(f_i \circ g^{-1}\right)^{(n)} (z)\, dz \right) dx \right] + \right. \tag{14}$$

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$$\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]\Bigg],$$

and

2)

$$\|E(f_1,...,f_r)(x_0)\| \leq \frac{1}{n!}$$

$$\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(a),g(x_0)]}(g(x_0)-g(a))^n\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]+\right.\right.$$

$$\left.\left.\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(x_0),g(b)]}(g(b)-g(x_0))^n\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]\right]\right\}. \tag{15}$$

Proof. Let $x_0\in[a,b]$ such that $\left(f_i\circ g^{-1}\right)^{(j)}(g(x_0))=0$, $j=1,...,n-1$; $i=1,...,r$. Let $x\in[a,x_0]$, then by Theorem 6 we have

$$f_i(x)-f_i(x_0)=\frac{1}{(n-1)!}\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz \tag{16}$$

$$=\frac{(-1)^n}{(n-1)!}\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz,$$

for $i=1,...,r$.

And for $x\in[x_0,b]$, then again by Theorem 6 we get

$$f_i(x)-f_i(x_0)=\frac{1}{(n-1)!}\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz, \tag{17}$$

for $i=1,...,r$.

We multiply (16) by $\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)$ to get:

$$\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)f_i(x)-\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)f_i(x_0)=$$

$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)(-1)^n}{(n-1)!}\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz, \tag{18}$$

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$\forall\ x \in [a, x_0]$; for $i = 1, ..., r$.

Similarly, we get (by (17))

$$\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x) - \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)}{(n-1)!} \int_{g(x_0)}^{g(x)} (g(x) - z)^{n-1} \left(f_i \circ g^{-1}\right)^{(n)}(z)\, dz, \tag{19}$$

$\forall\ x \in [x_0, b]$; for $i = 1, ..., r$.

Adding (18) and (19) as separate groups, we obtain

$$\sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x) - \sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{(-1)^n}{(n-1)!} \sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) \int_{g(x)}^{g(x_0)} (z - g(x))^{n-1} \left(f_i \circ g^{-1}\right)^{(n)}(z)\, dz, \tag{20}$$

$\forall\ x \in [a, x_0]$,

and

$$\sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x) - \sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{1}{(n-1)!} \sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) \int_{g(x_0)}^{g(x)} (g(x) - z)^{n-1} \left(f_i \circ g^{-1}\right)^{(n)}(z)\, dz, \tag{21}$$

$\forall\ x \in [x_0, b]$.

Next, we integrate (20) and (21) with respect to $x \in [a, b]$. We have

$$\sum_{i=1}^{r} \int_a^{x_0} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x)\, dx - \sum_{i=1}^{r} \left(\int_a^{x_0} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) dx\right) f_i(x_0) = \tag{22}$$

$$\frac{(-1)^n}{(n-1)!} \sum_{i=1}^{r} \left[\int_a^{x_0} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{n-1} \left(f_i \circ g^{-1}\right)^{(n)}(z)\, dz\right) dx\right],$$

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and

$$\sum_{i=1}^{r}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)f_i(x)\,dx-\sum_{i=1}^{r}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)dx\right)f_i(x_0)= \tag{23}$$

$$\frac{1}{(n-1)!}\sum_{i=1}^{r}\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right].$$

Finally, adding (22) and (23) we obtain the useful identity

$$E(f_1,...,f_r)(x_0):=$$

$$\sum_{i=1}^{r}\left[\int_{a}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)f_i(x)\,dx-\left(\int_{a}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)dx\right)f_i(x_0)\right]=\frac{1}{(n-1)!}$$

$$\sum_{i=1}^{r}\left[(-1)^n\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]+\right.$$

$$\left.\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]\right], \tag{24}$$

proving (14).

Therefore, we get that

$$\|E(f_1,...,f_r)(x_0)\|=$$

$$\left\|\sum_{i=1}^{r}\left[\int_{a}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)f_i(x)\,dx-\left(\int_{a}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)dx\right)f_i(x_0)\right]\right\|\leq\frac{1}{(n-1)!}$$

$$\left\{\sum_{i=1}^{r}\left[\left\|\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]\right\|\right.\right.$$

$$\left.\left.+\left\|\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]\right\|\right]\right\}\leq$$

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$$\frac{1}{(n-1)!}\left\{\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left\|\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)\right\|dx\right]\right.\right.$$

$$\left.\left.+\left[\int_{x_0}^{b}\left\|\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)\right\|dx\right]\right]\right\}\leq \tag{25}$$

$$\frac{1}{(n-1)!}\left\{\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left\|\left(f_i\circ g^{-1}\right)^{(n)}(z)\right\|dz\right)dx\right]\right.\right.$$

$$\left.\left.+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left\|\left(f_i\circ g^{-1}\right)^{(n)}(z)\right\|dz\right)dx\right]\right]\right\}=:(\xi)\,. \tag{26}$$

Hence it holds

$$\|E(f_1,...,f_r)(x_0)\|\leq(\xi)\,. \tag{27}$$

We have that

$$(\xi)\leq\frac{1}{n!}\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(a),g(x_0)]}\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)(g(x_0)-g(x))^n\,dx\right]\right.\right.$$

$$\left.\left.+\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)(g(x)-g(x_0))^n\,dx\right]\right]\right\}\leq \tag{28}$$

$$\frac{1}{n!}\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(a),g(x_0)]}(g(x_0)-g(a))^n\left(\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]\right.\right.$$

$$\left.\left.+\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(x_0),g(b)]}(g(b)-g(x_0))^n\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]\right]\right\}, \tag{29}$$

proving (15). ■

Next comes an L_1 estimate.

Theorem 8 *All as in Theorem 7. Then*

$$\|E(f_1,...,f_r)(x_0)\| \leq \frac{1}{(n-1)!}$$

$$\left\{ \sum_{i=1}^{r} \left[\left[\left\| \left\| \left(f_i \circ g^{-1}\right)^{(n)} \right\| \right\|_{L_1([g(a),g(x_0)])} \int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) (g(x_0) - g(x))^{n-1} dx \right] \right. \right.$$

$$\left. \left. + \left[\left\| \left\| \left(f_i \circ g^{-1}\right)^{(n)} \right\| \right\|_{L_1([g(x_0),g(b)])} \int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) (g(x) - g(x_0))^{n-1} dx \right] \right] \right\}. \tag{30}$$

Proof. By (26), (27), we get that

$$\|E(f_1,...,f_r)(x_0)\| \leq (\xi) \leq \frac{1}{(n-1)!}$$

$$\left\{ \sum_{i=1}^{r} \left[\left[\left\| \left\| \left(f_i \circ g^{-1}\right)^{(n)} \right\| \right\|_{L_1([g(a),g(x_0)])} \int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) (g(x_0) - g(x))^{n-1} dx \right] \right. \right.$$

$$\left. \left. + \left[\left\| \left\| \left(f_i \circ g^{-1}\right)^{(n)} \right\| \right\|_{L_1([g(x_0),g(b)])} \int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) (g(x) - g(x_0))^{n-1} dx \right] \right] \right\}, \tag{31}$$

proving (30). ■

An L_p estimate follows.

Theorem 9 *All as in Theorem 7, and let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\|E(f_1,...,f_r)(x_0)\| \leq \frac{1}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}}$$

$$\sum_{i=1}^{r} \left[\left\| \left\| \left(f_i \circ g^{-1}\right)^{(n)} \right\| \right\|_{L_q([g(a),g(x_0)])} \left(\int_a^{x_0} (g(x_0) - g(x))^{n-\frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) dx \right) \right.$$

$$\left. + \left\| \left\| \left(f_i \circ g^{-1}\right)^{(n)} \right\| \right\|_{L_q([g(x_0),g(b)])} \left(\int_{x_0}^{b} (g(x) - g(x_0))^{n-\frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) dx \right) \right]. \tag{32}$$

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Proof. By (26), (27), we get that

$$\|E(f_1,...,f_r)(x_0)\| \leq (\xi) \leq \frac{1}{(n-1)!}$$

$$\left\{\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{p(n-1)}dz\right)^{\frac{1}{p}}\right.\right.\right.$$

$$\left.\left(\int_{g(x)}^{g(x_0)}\left\|\left(f_i\circ g^{-1}\right)^{(n)}(z)\right\|^q dz\right)^{\frac{1}{q}}dx\right]+$$

$$\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{p(n-1)}dz\right)^{\frac{1}{p}}\right. \tag{33}$$

$$\left.\left.\left.\left(\int_{g(x_0)}^{g(x)}\left\|\left(f_i\circ g^{-1}\right)^{(n)}(z)\right\|^q dz\right)^{\frac{1}{q}}dx\right]\right]\right\} = \frac{1}{(n-1)!}$$

$$\left\{\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)\frac{(g(x_0)-g(x))^{\frac{p(n-1)|+1}{p}}}{(p(n-1)+1)^{\frac{1}{p}}}\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_q([g(a),g(x_0)])}dx\right]\right.\right.$$

$$\left.\left.+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)\frac{(g(x)-g(x_0))^{\frac{p(n-1)|+1}{p}}}{(p(n-1)+1)^{\frac{1}{p}}}\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_q([g(x_0),g(b)])}dx\right]\right]\right\}$$

$$= \frac{1}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}}$$

$$\left\{\sum_{i=1}^{r}\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_q([g(a),g(x_0)])}\left(\int_a^{x_0}(g(x_0)-g(x))^{n-\frac{1}{q}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right.\right.$$

$$\left.\left.+\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_q([g(x_0),g(b)])}\left(\int_{x_0}^{b}(g(x)-g(x_0))^{n-\frac{1}{q}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]\right\}, \tag{34}$$

proving (32). ■

Next we present a left generalized Opial type inequality for ordinary derivatives:

Theorem 10 *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, and $n\in\mathbb{N}$, $f\in C^{n}([a,b],A)$; where $[a,b]\subset\mathbb{R}$ and $(A,\|\cdot\|)$ is a Banach algebra. Let $g\in C^{1}([a,b])$, strictly increasing, such that $g^{-1}\in C^{n}([g(a),g(b)])$. We assume that $\left(f\circ g^{-1}\right)^{(j)}(g(x_0))=0$, $j=0,1,...,n-1$; where $x_0\in[a,b]$ be fixed. Then*

$$\int_{g(x_0)}^{g(x)}\left\|\left(f\circ g^{-1}\right)(z)\left(f\circ g^{-1}\right)^{(n)}(z)\right\|dz\leq$$

$$\frac{(g(x)-g(x_0))^{n+\frac{1}{p}-\frac{1}{q}}}{2^{\frac{1}{q}}(n-1)!\left[(p(n-1)+1)(p(n-1)+2)\right]^{\frac{1}{p}}}\left(\int_{g(x_0)}^{g(x)}\left\|\left(f\circ g^{-1}\right)^{(n)}(z)\right\|^{q}dz\right)^{\frac{2}{q}}, \tag{35}$$

for all $x_0\leq x\leq b$.

Proof. Let $x_0\in[a,b]$ such that $\left(f\circ g^{-1}\right)^{(j)}(g(x_0))=0$, $j=0,1,...,n-1$. For $x\in[x_0,b]$ by Theorem 6 we have

$$\left(f\circ g^{-1}\right)(g(z))=\frac{1}{(n-1)!}\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f\circ g^{-1}\right)^{(n)}(z)\,dz. \tag{36}$$

By Hölder's inequality we obtain

$$\left\|\left(f\circ g^{-1}\right)(g(x))\right\|\leq\frac{1}{(n-1)!}\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left\|\left(f\circ g^{-1}\right)^{(n)}(z)\right\|dz\leq \tag{37}$$

$$\frac{1}{(n-1)!}\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{p(n-1)}dt\right)^{\frac{1}{p}}\left(\int_{g(x_0)}^{g(x)}\left\|\left(f\circ g^{-1}\right)^{(n)}(z)\right\|^{q}dz\right)^{\frac{1}{q}}=$$

$$\frac{1}{(n-1)!}\frac{(g(x)-g(x_0))^{\frac{p(n-1)+1}{p}}}{(p(n-1)+1)^{\frac{1}{p}}}\left(\int_{g(x_0)}^{g(x)}\left\|\left(f\circ g^{-1}\right)^{(n)}(z)\right\|^{q}dz\right)^{\frac{1}{q}}.$$

Call

$$\varphi(g(x)):=\int_{g(x_0)}^{g(x)}\left\|\left(f\circ g^{-1}\right)^{(n)}(z)\right\|^{q}dz, \tag{38}$$

$\varphi(g(x_0))=0$.

Thus

$$\frac{d\varphi(g(x))}{dg(x)}=\left\|\left(f\circ g^{-1}\right)^{(n)}(g(x))\right\|^{q}\geq0, \tag{39}$$

and

$$\left(\frac{d\varphi(g(x))}{dg(x)}\right)^{\frac{1}{q}}=\left\|\left(f\circ g^{-1}\right)^{(n)}(g(x))\right\|\geq0, \tag{40}$$

$\forall\ g(x)\in[g(x_0),g(b)]$.

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Consequently, we get

$$\left\| \left(f \circ g^{-1} \right) \left(g \left(w \right) \right) \right\| \left\| \left(f \circ g^{-1} \right)^{(n)} \left(g \left(w \right) \right) \right\| \leq$$

$$\frac{\left(g \left(w \right) - g \left(x_0 \right) \right)^{\frac{p(n-1)+1}{p}}}{\left(n-1 \right)! \left(p \left(n-1 \right) + 1 \right)^{\frac{1}{p}}} \left(\varphi \left(g \left(w \right) \right) \frac{d\varphi \left(g \left(w \right) \right)}{dg \left(w \right)} \right)^{\frac{1}{q}}, \tag{41}$$

$\forall \ g \left(w \right) \in \left[g \left(x_0 \right), g \left(b \right) \right].$

Then we observe that

$$\int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1} \right) \left(g \left(w \right) \right) \left(f \circ g^{-1} \right)^{(n)} \left(g \left(w \right) \right) \right\| dg \left(w \right) \overset{(7)}{\leq}$$

$$\int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1} \right) \left(g \left(w \right) \right) \right\| \left\| \left(f \circ g^{-1} \right)^{(n)} \left(g \left(w \right) \right) \right\| dg \left(w \right) \leq$$

$$\frac{1}{\left(n-1 \right)! \left(p \left(n-1 \right) + 1 \right)^{\frac{1}{p}}}$$

$$\int_{g(x_0)}^{g(x)} \left(g \left(w \right) - g \left(x_0 \right) \right)^{\frac{p(n-1)+1}{p}} \left(\varphi \left(g \left(w \right) \right) \frac{d\varphi \left(g \left(w \right) \right)}{dg \left(w \right)} \right)^{\frac{1}{q}} dg \left(w \right) \leq \tag{42}$$

$$\frac{1}{\left(n-1 \right)! \left(p \left(n-1 \right) + 1 \right)^{\frac{1}{p}}}$$

$$\left(\int_{g(x_0)}^{g(x)} \left(g \left(w \right) - g \left(x_0 \right) \right)^{p(n-1)+1} dg \left(w \right) \right)^{\frac{1}{p}} \left(\int_{g(x_0)}^{g(x)} \varphi \left(g \left(w \right) \right) \frac{d\varphi \left(g \left(w \right) \right)}{dg \left(w \right)} dg \left(w \right) \right)^{\frac{1}{q}} =$$

$$\frac{1}{\left(n-1 \right)! \left(p \left(n-1 \right) + 1 \right)^{\frac{1}{p}} \left(p \left(n-1 \right) + 2 \right)^{\frac{1}{p}}}$$

$$\left(g \left(x \right) - g \left(x_0 \right) \right)^{\frac{p(n-1)+2}{p}} \left(\int_{g(x_0)}^{g(x)} \varphi \left(g \left(w \right) \right) \ d\varphi \left(g \left(w \right) \right) \right)^{\frac{1}{q}} = \tag{43}$$

$$\frac{\left(g \left(x \right) - g \left(x_0 \right) \right)^{n + \frac{1}{p} - \frac{1}{q}}}{\left(n-1 \right)! \left(p \left(n-1 \right) + 1 \right)^{\frac{1}{p}} \left(p \left(n-1 \right) + 2 \right)^{\frac{1}{p}}} \left(\frac{\varphi^2 \left(g \left(x \right) \right)}{2} \right)^{\frac{1}{q}} =$$

$$\frac{\left(g \left(x \right) - g \left(x_0 \right) \right)^{n + \frac{1}{p} - \frac{1}{q}}}{2^{\frac{1}{q}} \left(n-1 \right)! \left(\left(p \left(n-1 \right) + 1 \right) \left(p \left(n-1 \right) + 2 \right) \right)^{\frac{1}{p}}} \left(\int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1} \right)^{(n)} \left(z \right) \right\|^q dz \right)^{\frac{2}{q}}, \tag{44}$$

for all $g \left(x_0 \right) \leq g \left(x \right) \leq g \left(b \right)$, proving (35). ■

The corresponding right generalized Opial type inequality follows:

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Theorem 11 *All as in Theorem 10. Then*

$$\int_{g(x)}^{g(x_0)} \left\| \left(f \circ g^{-1}\right)(z) \left(f \circ g^{-1}\right)^{(n)}(z) \right\| dz \leq$$

$$\frac{\left(g\left(x_0\right)-g\left(x\right)\right)^{n+\frac{1}{p}-\frac{1}{q}}}{2^{\frac{1}{q}}\left(n-1\right)!\left(\left(p\left(n-1\right)+1\right)\left(p\left(n-1\right)+2\right)\right)^{\frac{1}{p}}} \left(\int_{g(x)}^{g(x_0)} \left\| \left(f \circ g^{-1}\right)^{(n)}(z) \right\|^q dz\right)^{\frac{2}{q}}, \tag{45}$$

for all $a \leq x \leq x_0$.

Proof. As similar to Theorem 10 is omitted. ■

Next we present a left generalized Hilbert-Pachpatte inequality for ordinary derivatives.

Theorem 12 *Let $i = 1, 2$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $n_i \in \mathbb{N}$, $f_i \in C^{n_i}\left(\left[a_i, b_i\right], A\right)$; where $\left[a_i, b_i\right] \subset \mathbb{R}$ and $\left(A, \|\cdot\|\right)$ is a Banach algebra. Let $g_i \in C^1\left(\left[a_i, b_i\right]\right)$, strictly increasing, such that $g_i^{-1} \in C^{n_i}\left(\left[g_i\left(a_i\right), g_i\left(b_i\right)\right]\right)$. We assume that $\left(f_i \circ g_i^{-1}\right)^{(j_i)}\left(g_i\left(x_{0i}\right)\right) = 0$, $j_i = 0, 1, ..., n_i - 1$; where $x_{0i} \in \left[a_i, b_i\right]$ be fixed. Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)} \int_{g_2(x_{02})}^{g_2(b_2)} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(z_1 - g_1(x_{01}))^{p(n_1-1)+1}}{p(p(n_1-1)+1)} + \frac{(z_2 - g_2(x_{02}))^{q(n_2-1)+1}}{q(q(n_2-1)+1)}\right)} \leq$$

$$\frac{\left(g_1\left(b_1\right)-g_1\left(x_{01}\right)\right)\left(g_2\left(b_2\right)-g_2\left(x_{02}\right)\right)}{\left(n_1-1\right)!\left(n_2-1\right)!} \tag{46}$$

$$\left\| \left\| \left(f_1 \circ g_1^{-1}\right)^{(n_1)} \right\| \right\|_{L_q([g_1(x_{01}), g_1(b_1)], A)} \left\| \left\| \left(f_2 \circ g_2^{-1}\right)^{(n_2)} \right\| \right\|_{L_p([g_2(x_{02}), g_2(b_2)], A)}.$$

Proof. Let $i = 1, 2$; $x_0 \in \left[a_i, b_i\right]$, such that $\left(f_i \circ g_i^{-1}\right)^{(j_i)}\left(g_i\left(x_{0i}\right)\right) = 0$, $j_i = 0, 1, ..., n_i - 1$.

For $x_i \in \left[x_{0i}, b_i\right]$ by Theorem 6 we have

$$\left(f_i \circ g_i^{-1}\right)\left(g_i\left(x_i\right)\right) = \frac{1}{\left(n_i-1\right)!} \int_{g_i(x_{0i})}^{g_i(x_i)} \left(g_i\left(x_i\right)-z_i\right)^{n_i-1} \left(f_i \circ g_i^{-1}\right)^{(n_i)}(z_i)\, dz_i. \tag{47}$$

As in (37) we have

$$\left\| \left(f_1 \circ g_1^{-1}\right)\left(g_1\left(x_1\right)\right) \right\| \leq \frac{1}{\left(n_1-1\right)!} \frac{\left(g_1\left(x_1\right)-g_1\left(x_{01}\right)\right)^{\frac{p(n_1-1)+1}{p}}}{\left(p\left(n_1-1\right)+1\right)^{\frac{1}{p}}}$$

$$\left(\int_{g_1(x_{01})}^{g_1(x_1)} \left\| \left(f_1 \circ g_1^{-1}\right)^{(n_1)}(z) \right\|^q dz\right)^{\frac{1}{q}} \leq$$

$$\frac{1}{(n_1-1)!}\frac{(g_1(x_1)-g_1(x_{01}))^{\frac{p(n_1-1)+1}{p}}}{(p(n_1-1)+1)^{\frac{1}{p}}}\left\|\left\|\left(f_1\circ g_1^{-1}\right)^{(n_1)}\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)])}, \tag{48}$$

for all $x_1 \in [x_{01}, b_1]$.

Similarly, we obtain that

$$\left\|\left(f_2\circ g_2^{-1}\right)(g_2(x_2))\right\| \leq \frac{1}{(n_2-1)!}\frac{(g_2(x_2)-g_2(x_{02}))^{\frac{q(n_2-1)+1}{q}}}{(q(n_2-1)+1)^{\frac{1}{q}}}$$

$$\left\|\left\|\left(f_2\circ g_2^{-1}\right)^{(n_2)}\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)])}, \tag{49}$$

for all $x_2 \in [x_{02}, b_2]$.

By (48) and (49) we get

$$\left\|\left(f_1\circ g_1^{-1}\right)(g_1(x_1))\left(f_2\circ g_2^{-1}\right)(g_2(x_2))\right\| \leq$$

$$\left\|\left(f_1\circ g_1^{-1}\right)(g_1(x_1))\right\|\left\|\left(f_2\circ g_2^{-1}\right)(g_2(x_2))\right\| \leq \frac{1}{(n_1-1)!(n_2-1)!}$$

$$\frac{(g_1(x_1)-g_1(x_{01}))^{\frac{p(n_1-1)+1}{p}}}{(p(n_1-1)+1)^{\frac{1}{p}}}\frac{(g_2(x_2)-g_2(x_{02}))^{\frac{q(n_2-1)+1}{q}}}{(q(n_2-1)+1)^{\frac{1}{q}}} \tag{50}$$

$$\left\|\left\|\left(f_1\circ g_1^{-1}\right)^{(n_1)}\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)])}\left\|\left\|\left(f_2\circ g_2^{-1}\right)^{(n_2)}\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)])} \leq$$

(using Young's inequality for $a, b \geq 0$, $a^{\frac{1}{p}}b^{\frac{1}{q}} \leq \frac{a}{p}+\frac{b}{q}$)

$$\frac{1}{(n_1-1)!(n_2-1)!}\left(\frac{(g_1(x_1)-g_1(x_{01}))^{p(n_1-1)+1}}{p(p(n_1-1)+1)}+\frac{(g_2(x_2)-g_2(x_{02}))^{q(n_2-1)+1}}{q(q(n_2-1)+1)}\right) \tag{51}$$

$$\left\|\left\|\left(f_1\circ g_1^{-1}\right)^{(n_1)}\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)])}\left\|\left\|\left(f_2\circ g_2^{-1}\right)^{(n_2)}\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)])},$$

$\forall\ (x_1, x_2) \in [x_{01}, b_1] \times [x_{02}, b_2]$.

So far we have

$$\frac{\left\|\left(f_1\circ g_1^{-1}\right)(g_1(x_1))\left(f_2\circ g_2^{-1}\right)(g_2(x_2))\right\|}{\left(\frac{(g_1(x_1)-g_1(x_{01}))^{p(n_1-1)+1}}{p(p(n_1-1)+1)}+\frac{(g_2(x_2)-g_2(x_{02}))^{q(n_2-1)+1}}{q(q(n_2-1)+1)}\right)} \leq \tag{52}$$

$$\frac{1}{(n_1-1)!(n_2-1)!}\left\|\left\|\left(f_1\circ g_1^{-1}\right)^{(n_1)}\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)],A)}$$

$$\left\|\left\|\left(f_2\circ g_2^{-1}\right)^{(n_2)}\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)],A)},$$

$\forall\ (x_1, x_2) \in [x_{01}, b_1] \times [x_{02}, b_2]$.

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The denominator in (52) can be zero, only when both $g_1(x_1) = g_1(x_{01})$ and $g_2(x_2) = g_2(x_{02})$.

Therefore we obtain (46), by integrating (52) over $[g_1(x_{01}), g_1(b_1)] \times [g_2(x_{02}), g_2(b_2)]$. ■

It follows the right generalized Hilbert-Pachpate inequality for ordinary derivatives.

Theorem 13 *All as in Theorem 12. Then*

$$\int_{g_1(a_1)}^{g_1(x_{01})} \int_{g_2(a_2)}^{g_2(x_{02})} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(g_1(x_{01}) - z_1)^{p(n_1-1)+1}}{p(p(n_1-1)+1)} + \frac{(g_2(x_{02}) - z_2)^{q(n_2-1)+1}}{q(q(n_2-1)+1)} \right)} \leq$$

$$\frac{(g_1(x_{01}) - g_1(a_1))(g_2(x_{02}) - g_2(a_2))}{(n_1-1)!(n_2-1)!} \tag{53}$$

$$\left\| \left\| \left(f_1 \circ g_1^{-1}\right)^{(n_1)} \right\| \right\|_{L_q([g_1(a_1), g_1(x_{01})], A)} \left\| \left\| \left(f_2 \circ g_2^{-1}\right)^{(n_2)} \right\| \right\|_{L_p([g_2(a_2), g_2(x_{02})], A)}.$$

Proof. As similar to theorem 12 is omitted. ■

5 Applications

We make

Remark 14 *Assume next that $(A, \|\cdot\|)$ is a commutative Banach algebra. Then, we get that*

$$E(f_1, ..., f_r)(x_0) \overset{(13)}{=} r \int_a^b \left(\prod_{j=1}^r f_j(x) \right) dx - \sum_{i=1}^r \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) dx \right) f_i(x_0), \tag{54}$$

$x_0 \in [a, b]$.

When $r = 2$, we have that

$$E(f_1, f_2)(x_0) = 2 \int_a^b f_1(x) f_2(x) dx - f_1(x_0) \int_a^b f_2(x) dx - f_2(x_0) \int_a^b f_1(x) dx, \tag{55}$$

$x_0 \in [a, b]$.

We give

Corollary 15 *(to Theorem 7) All as in Theorem 7, $(A, \|\cdot\|)$ is a commutative Banach algebra, $r = 2$. Then*

$$\|E(f_1, f_2)(x_0)\| \leq \frac{1}{n!} \sum_{i=1}^2 \left[\left[\left\| \left\| \left(f_i \circ g^{-1}\right)^{(n)} \right\| \right\|_{\infty, [g(a), g(x_0)]} \right. \right.$$

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$$\left(g\left(x_{0}\right)-g\left(a\right)\right)^{n}\left(\int_{a}^{x_{0}}\left(\prod_{\substack{j=1\\j\neq i}}^{2}\left\|f_{j}\left(x\right)\right\|\right)dx\right)\Bigg]+$$

$$\left[\left\|\left\|\left(f_{i}\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(x_{0}),g(b)]}\left(g\left(b\right)-g\left(x_{0}\right)\right)^{n}\left(\int_{x_{0}}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{2}\left\|f_{j}\left(x\right)\right\|\right)dx\right)\right]\Bigg]. \tag{56}$$

It follows

Corollary 16 *(to Corollary 15) All as in Corollary 15, with* $g\left(t\right)=e^{t}$*. Then*

$$\left\|E\left(f_{1},f_{2}\right)\left(x_{0}\right)\right\|\leq\frac{1}{n!}\sum_{i=1}^{2}\Bigg[\Bigg[\left\|\left\|\left(f_{i}\circ\log\right)^{(n)}\right\|\right\|_{\infty,[e^{a},e^{x_{0}}]}$$

$$\left(e^{x_{0}}-e^{a}\right)^{n}\left(\int_{a}^{x_{0}}\left(\prod_{\substack{j=1\\j\neq i}}^{2}\left\|f_{j}\left(x\right)\right\|\right)dx\right)\Bigg]+$$

$$\left[\left\|\left\|\left(f_{i}\circ\log\right)^{(n)}\right\|\right\|_{\infty,[e^{x_{0}},e^{b}]}\left(e^{b}-e^{x_{0}}\right)^{n}\left(\int_{x_{0}}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{2}\left\|f_{j}\left(x\right)\right\|\right)dx\right)\right]\Bigg]. \tag{57}$$

We continue with

Corollary 17 *(to Theorem 10) All as in Theorem 10 for* $g\left(t\right)=e^{t}$*. Then*

$$\int_{e^{x_{0}}}^{e^{x}}\left\|\left(\left(f\circ\log\right)\left(z\right)\right)\left(f\circ\log\right)^{(n)}\left(z\right)\right\|dz\leq$$

$$\frac{\left(e^{x}-e^{x_{0}}\right)^{n+\frac{1}{p}-\frac{1}{q}}}{2^{\frac{1}{q}}\left(n-1\right)!\left[\left(p\left(n-1\right)+1\right)\left(p\left(n-1\right)+2\right)\right]^{\frac{1}{p}}}\left(\int_{e^{x_{0}}}^{z}\left\|\left(f\circ\log\right)^{(n)}\left(z\right)\right\|^{q}dz\right)^{\frac{2}{q}}, \tag{58}$$

for all $x_{0}\leq x\leq b$*.*

We finish with

Corollary 18 *(to Theorem 12) All as in Theorem 12 for* $g_{i}\left(t\right)=e^{t}$*,* $i=1,2$*. Then*

$$\int_{e^{x_{01}}}^{e^{b_{1}}}\int_{e^{x_{02}}}^{e^{b_{2}}}\frac{\left\|\left(f_{1}\circ\log\right)\left(z_{1}\right)\left(f_{2}\circ\log\right)\left(z_{2}\right)\right\|dz_{1}dz_{2}}{\left(\frac{\left(z_{1}-e^{x_{01}}\right)^{p(n_{1}-1)+1}}{p(p(n_{1}-1)+1)}+\frac{\left(z_{2}-e^{x_{02}}\right)^{q(n_{2}-1)+1}}{q(q(n_{2}-1)+1)}\right)}\leq$$

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$$\frac{\left(e^{b_1}-e^{x_{01}}\right)\left(e^{b_2}-e^{x_{02}}\right)}{(n_1-1)!\,(n_2-1)!} \tag{59}$$
$$\left\|\left\|\left(f_1\circ\log\right)^{(n_1)}\right\|\right\|_{L_q\left(\left[e^{x_{01}},e^{b_1}\right],A\right)}\left\|\left\|\left(f_2\circ\log\right)^{(n_2)}\right\|\right\|_{L_p\left(\left[e^{x_{02}},e^{b_2}\right],A\right)}.$$

The simplest applications derive when $g(t)=t$ and $A=\mathbb{R}$, leading to basic known results.

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Multivariate Ostrowski type inequalities for several Banach algebra valued functions

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Abstract

Here we are dealing with several smooth functions from a compact convex set of $\mathbb{R}^k$, $k \geq 2$ to a Banach algebra. For these we prove general multivariate Ostrowski type inequalities with estimates in norms $\|\cdot\|_p$, for all $1 \leq p \leq \infty$. We provide also interesting applications.

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1 Introduction

In 1938, A Ostrowski [5] proved the following famous inequality:

Theorem 1 *(1938, Ostrowski [6]) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|f'\|_\infty^{\sup} := \sup_{t \in (a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty^{\sup}, \tag{1}$$

for any $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

Since then there has been a lot of activity around these inequalities with important applications to Numerical Analysis and Probability.

This article is also greatly motivated by the following result:

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Theorem 2 *(see [1]) Let* $f \in C^1\left(\prod_{i=1}^{k}[a_i,b_i]\right)$, *where* $a_i < b_i$; $a_i, b_i \in \mathbb{R}$, $i = 1,...,k$, *ans let* $\overrightarrow{x_0} := (x_{01},...,x_{0k}) \in \prod_{i=1}^{k}[a_i,b_i]$ *be fixed. Then*

$$\left|\frac{1}{\prod_{i=1}^{k}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_i}^{b_i}...\int_{a_k}^{b_k} f(z_1,...,z_k)\,dz_1...dz_k - f(\overrightarrow{x_0})\right| \leq \tag{2}$$

$$\sum_{i=1}^{k}\left(\frac{(x_{0i}-a_i)^2+(b_i-x_{0i})^2}{2(b_i-a_i)}\right)\left\|\frac{\partial f}{\partial z_i}\right\|_{\infty}.$$

Inequality (2) is sharp, here the optimal function is

$$f^*(z_1,...,z_k) := \sum_{i=1}^{k}|z_i-x_{0i}|^{\alpha_i}, \ \ \alpha_i > 1.$$

Clearly inequality (2) generalizes inequality (1) to multidimension.

We are inspired also by [2].

In this article we establish multivariate Ostrowski type inequalities for several smooth functions from a compact convex subset of $\mathbb{R}^k$, $k \geq 2$, to a Banach algebra. These involve the norms $\|\cdot\|_p$, $1 \leq p \leq \infty$.

2 About Banach Algebras

All here come from [6].

We need

Definition 3 *([6], p. 245) A complex algebra is a vector space A over the complex field $\mathbb{C}$ in which a multiplication is defined that satisfies*

$$x(yz) = (xy)z, \tag{3}$$

$$(x+y)z = xz+yz, \ \ x(y+z) = xy+xz, \tag{4}$$

and

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \tag{5}$$

for all x, y and z in A and for all scalars α.

Additionally if A is a Banach space with respect to a norm that satisfies the multiplicative inequality

$$\|xy\| \leq \|x\|\|y\| \ \ (x \in A,\ y \in A) \tag{6}$$

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and if A contains a unit element e such that

$$xe = ex = x \quad (x \in A) \tag{7}$$

and

$$\|e\| = 1, \tag{8}$$

then A is called a Banach algebra.

A is commutative iff $xy = yx$ for all $x, y \in A$.

We make

Remark 4 *Commutativity of A will be explicited stated when needed.*

There exists at most one $e \in A$ that satisfies (7).

Inequality (6) makes multiplication to be continuous, more precisely left and right continuous, see [6], p. 246.

Multiplication in A is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.

For nice examples about Banach algebras see [6], p. 247-248, § 10.3.

We also make

Remark 5 *Next we mention about integration of A-valued functions, see [6], p. 259, § 10.22:*

If A is a Banach algebra and f is a continuous A-valued function on some compact Hausdorff space Q on which a complex Borel measure μ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [6], simply because A is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then

$$x \int_Q f \ d\mu = \int_Q x f(p) \ d\mu(p) \tag{9}$$

and

$$\left(\int_Q f \ d\mu \right) x = \int_Q f(p) \, x \ d\mu(p) \,. \tag{10}$$

The vector integrals we will involve in our article follow (9) and (10).

3 Vector Analysis Background

(see [8], pp. 83-94)

Let $f(t)$ be a function defined on $[a,b] \subseteq \mathbb{R}$ taking values in a real or complex normed linear space $(X, \|\cdot\|)$, Then $f(t)$ is said to be differentiable at a point $t_0 \in [a,b]$ if the limit

$$f'(t_0) = \lim_{h \to 0} \frac{f(t_0+h) - f(t_0)}{h} \tag{11}$$

exists in X, the convergence is in $\|\cdot\|$. This is called the derivative of $f(t)$ at $t = t_0$.

We call $f(t)$ differentiable on $[a,b]$, iff there exists $f'(t) \in X$ for all $t \in [a,b]$.

Similarly and inductively are defined higher order derivatives of f, denoted $f'', f^{(3)}, ..., f^{(k)}$, $k \in \mathbb{N}$, just as for numerical functions.

For all the properties of derivatives see [8], pp. 83-86.

Let now $(X, \|\cdot\|)$ be a Banach space, and $f : [a,b] \to X$.

We define the vector valued Riemann integral $\int_a^b f(t)\,dt \in X$ as the limit of the vector valued Riemann sums in X, convergence is in $\|\cdot\|$. The definition is as for the numerical valued functions.

If $\int_a^b f(t)\,dt \in X$ we call f integrable on $[a,b]$. If $f \in C([a,b], X)$, then f is integrable, [8], p. 87.

For all the properties of vector valued Riemann integrals see [8], pp. 86-91.

We define the space $C^n([a,b], X)$, $n \in \mathbb{N}$, of n-times continuousky differentiable functions from $[a,b]$ into X; here continuity is with respect to $\|\cdot\|$ and defined in the usual way as for numerical functions·.

Let $(X, \|\cdot\|)$ be a Banach space and $f \in C^n([a,b], X)$, then we have the vector valued Taylor's formula, see [8], pp. 93-94, and also [7], (IV, 9; 47).

It holds

$$\begin{gathered} f(y) - f(x) - f'(x)(y-x) - \frac{1}{2} f''(x)(y-x)^2 - ... - \frac{1}{(n-1)!} f^{(n-1)}(x)(y-x)^{n-1} \\ = \frac{1}{(n-1)!} \int_x^y (y-t)^{n-1} f^{(n)}(t)\,dt, \quad \forall\, x, y \in [a,b]. \end{gathered} \tag{12}$$

In particular (12) is true when $X = \mathbb{R}^m, \mathbb{C}^m$, $m \in \mathbb{N}$, etc.

A function $f(t)$ with values in a normed linear space X is said to be piecewise continuous (see [8], p. 85) on the interval $a \leq t \leq b$ if there exists a partition $a = t_0 < t_1 < t_2 < ... < t_n = b$ such that $f(t)$ is continuous on every open interval $t_k < t < t_{k+1}$ and has finite limits $f(t_0+0)$, $f(t_1-0)$, $f(t_1+0)$, $f(t_2-0)$, $f(t_2+0)$, ..., $f(t_n-0)$.

Here $f(t_k - 0) = \lim_{t \uparrow t_k} f(t)$, $f(t_k+0) = \lim_{t \downarrow t_k} f(t)$.

The values of $f(t)$ at the points t_k can be arbitrary or even undefined.

A function $f(t)$ with values in normed linear space X is said to be piecewise smooth on $[a,b]$, if it is continuous on $[a,b]$ and has a derivative $f'(t)$ at all but a finite number of points of $[a,b]$, and if $f'(t)$ is piecewise continuous on $[a,b]$ (see [8], p. 85).

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Let $u(t)$ and $v(t)$ be two piecewise smooth functions on $[a,b]$, one a numerical function and the other a vector function with values in Banach space X. Then we have the following integration by parts formula

$$\int_a^b u(t)\,dv(t) = u(t)\,v(t)\,|_a^b - \int_a^b v(t)\,du(t)\,, \tag{13}$$

see [8], p. 93.

We mention also the mean value theorem for Banach space valued functions.

Theorem 6 *(see [4], p. 3) Let $f \in C([a,b],X)$, where X is a Banach space. Assume f' exists on $[a,b]$ and $\|f'(t)\| \leq K$, $a < t < b$, then*

$$\|f(b) - f(a)\| \leq K(b-a)\,. \tag{14}$$

Here the multiple Riemann integral of a function from a real box or a real compact and convex subset to a Banach space is defined similarly to numerical one however convergence is with respect to $\|\cdot\|$. Similarly are defined the vector valued partial derivatives as in the numerical case.

We mention the equality of vector valued mixed partiasl derivatives.

Proposition 7 *(see Proposition 4.11 of [3], p. 90) Let $Q = (a,b)\times(c,d) \subseteq \mathbb{R}^2$ and $f \in C(Q,X)$, where $(X,\|\cdot\|)$ is a Banach space. Assume that $\frac{\partial}{\partial t} f(s,t)$, $\frac{\partial}{\partial s} f(s,t)$ and $\frac{\partial^2}{\partial t \partial s} f(s,t)$ exist and are continuous for $(s,t) \in Q$, then $\frac{\partial^2}{\partial s \partial t} f(s,t)$ exists for $(s,t) \in Q$ and*

$$\frac{\partial^2}{\partial s \partial t} f(s,t) = \frac{\partial^2}{\partial t \partial s} f(s,t)\,, \quad \textit{for } (s,t) \in Q. \tag{15}$$

4 Main Results

We present general Ostrowski type inequalities results regarding several Banach algebra valued functions.

Theorem 8 *Let $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; $(A,\|\cdot\|)$ a Banach algebra and $f_i \in C^{n+1}(Q,A)$, $i = 1,...,r$; $r \in \mathbb{N}$, $n \in \mathbb{Z}_+$, and fixed $\overrightarrow{x_0} \in Q \subset \mathbb{R}^k$, $k \geq 2$, where Q is a compact and convex subset. Here all vector partial derivatives $f_{i\alpha} := \frac{\partial^\alpha f_i}{\partial z^\alpha}$, where $\alpha = (\alpha_1,...,\alpha_k)$, $\alpha_\lambda \in \mathbb{Z}^+$, $\lambda = 1,...,k$, $|\alpha| = \sum_{\lambda=1}^k \alpha_\lambda = j$, $j = 1,...,n$, fulfill $f_{i\alpha}(\overrightarrow{x_0}) = 0$, $i = 1,...,r$.*

Denote

$$D_{n+1}(f_i) := \max_{\alpha:|\alpha|=n+1} \|\|f_{i\alpha}\|\|_{\infty,Q}\,, \tag{16}$$

$i = 1,...,r$, and

$$\|\overrightarrow{z} - \overrightarrow{x_0}\|_{l_1} := \sum_{\lambda=1}^k |z_\lambda - x_{0\lambda}|\,. \tag{17}$$

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Then

$$\left\| \sum_{i=1}^{r} \int_{Q} \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_{\rho}\left(\overrightarrow{z}\right) \right) f_{i}\left(\overrightarrow{z}\right) d\overrightarrow{z} - \sum_{i=1}^{r} \left(\int_{Q} \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_{\rho}\left(\overrightarrow{z}\right) \right) d\overrightarrow{z} \right) f_{i}\left(\overrightarrow{x_0}\right) \right\| \leq \tag{18}$$

$$\frac{\max\limits_{i \in \{1,\dots,r\}} D_{n+1}\left(f_i\right)}{(n+1)!} \sum_{i=1}^{r} \left(\int_{Q} \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| f_{\rho}\left(\overrightarrow{z}\right) \right\| \right) \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1} d\overrightarrow{z} \right) \leq$$

$$\frac{\max\limits_{i \in \{1,\dots,r\}} D_{n+1}\left(f_i\right)}{(n+1)!} \min \left\{ \left(\int_{Q} \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1} d\overrightarrow{z} \right) \left[\sum_{i=1}^{r} \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| \left\| f_{\rho} \right\| \right\|_{\infty,Q} \right) \right], \right.$$

$$\left\| \left\| \cdot - \overrightarrow{x_0} \right\|_{l_1}^{n+1} \right\|_{\infty,Q} \left[\sum_{i=1}^{r} \left\| \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| f_{\rho} \right\| \right) \right\|_{L_1(Q,A)} \right],$$

$$\left. \left\| \left\| \cdot - \overrightarrow{x_0} \right\|_{l_1}^{n+1} \right\|_{L_p(Q,A)} \left[\sum_{i=1}^{r} \left[\left\| \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| f_{\rho} \right\| \right) \right\|_{L_q(Q,A)} \right] \right] \right\}. \tag{19}$$

Proof. Take $g_{i\overrightarrow{z}}(t) := f_i\left(\overrightarrow{x_0} + t\left(\overrightarrow{z} - \overrightarrow{x_0}\right)\right)$, $0 \leq t \leq 1$; $i = 1, \dots, r$. Notice that $g_{i\overrightarrow{z}}(0) = f_i\left(\overrightarrow{x_0}\right)$ and $g_{i\overrightarrow{z}}(1) = f_i\left(\overrightarrow{z}\right)$. The jth derivative of $g_{i\overrightarrow{z}}(t)$, based on Proposition 7, is given by

$$g_{i\overrightarrow{z}}^{(j)}(t) = \left[\left(\sum_{\lambda=1}^{k} \left(z_{\lambda} - x_{0\lambda}\right) \frac{\partial}{\partial z_{\lambda}} \right)^{j} f_i \right] \left(x_{01} + t\left(z_1 - x_{01}\right), \dots, x_{0k} + t\left(z_k - x_{0k}\right)\right) \tag{20}$$

and

$$g_{i\overrightarrow{z}}^{(j)}(0) = \left[\left(\sum_{\lambda=1}^{k} \left(z_{\lambda} - x_{0\lambda}\right) \frac{\partial}{\partial z_{\lambda}} \right)^{j} f_i \right] \left(\overrightarrow{x_0}\right), \tag{21}$$

for $j = 1, \dots, n+1$; $i = 1, \dots, r$.

Let $f_{i\alpha}$ be a partial derivative of $f_i \in C^{n+1}(Q, A)$. Because by assumption of the theorem we have $f_{i\alpha}\left(\overrightarrow{x_0}\right) = 0$ for all $\alpha : |\alpha| = j$, $j = 1, \dots, n$, we find that

$$g_{i\overrightarrow{z}}^{(j)}(0) = 0, \quad j = 1, \dots, n; \; i = 1, \dots, r.$$

Hence by vector Taylor's theorem (12) we see that

$$f_i\left(\overrightarrow{z}\right) - f_i\left(\overrightarrow{x_0}\right) = \sum_{j=1}^{n} \frac{g_{i\overrightarrow{z}}^{(j)}(0)}{j!} + R_{in}\left(\overrightarrow{z}, 0\right) = R_{in}\left(\overrightarrow{z}, 0\right), \tag{22}$$

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where

$$R_{in}\left(\overrightarrow{z},0\right):=\int_0^1\left(\int_0^{t_1}...\left(\int_0^{t_{n-1}}\left(g_{i\overrightarrow{z}}^{(n)}\left(t_n\right)-g_{i\overrightarrow{z}}^{(n)}\left(0\right)\right)dt_n\right)...\right)dt_1, \quad (23)$$

$i=1,...,r.$

Therefore,

$$\left\|R_{in}\left(\overrightarrow{z},0\right)\right\|\leq\int_0^1\left(\int_0^{t_1}...\left(\int_0^{t_{n-1}}\left\|\left\|g_{i\overrightarrow{z}}^{(n+1)}\left(\xi\left(t_n\right)\right)\right\|\right\|_{\infty}t_ndt_n\right)...\right)dt_1, \quad (24)$$

by the vector mean value Theorem 6 applied on $g_{i\overrightarrow{z}}^{(n)}$ over $(0,t_n)$. Moreover, we get

$$\left\|R_{in}\left(\overrightarrow{z},0\right)\right\|\leq\left\|\left\|g_{i\overrightarrow{z}}^{(n+1)}\right\|\right\|_{\infty,[0,1]}\int_0^1\int_0^{t_1}...\left(\int_0^{t_{n-1}}t_ndt_n\right)...dt_1$$

$$=\frac{\left\|\left\|g_{i\overrightarrow{z}}^{(n+1)}\right\|\right\|_{\infty,[0,1]}}{(n+1)!}. \quad (25)$$

However, there exists a $t_{i0}\in[0,1]$ such that $\left\|\left\|g_{i\overrightarrow{z}}^{(n+1)}\right\|\right\|_{\infty,[0,1]}=\left\|g_{i\overrightarrow{z}}^{(n+1)}\left(t_{i0}\right)\right\|$.

That is

$$\left\|\left\|g_{i\overrightarrow{z}}^{(n+1)}\right\|\right\|_{\infty,[0,1]}=\left\|\left[\left(\sum_{\lambda=1}^k\left(z_\lambda-x_{0\lambda}\right)\frac{\partial}{\partial z_\lambda}\right)^{n+1}f_i\right]\left(\overrightarrow{x_0}+t_{i0}\left(\overrightarrow{z}-\overrightarrow{z_{0i}}\right)\right)\right\|$$

$$\leq\left[\left(\sum_{\lambda=1}^k\left|z_\lambda-x_{0\lambda}\right|\left\|\frac{\partial}{\partial z_\lambda}\right\|\right)^{n+1}f_i\right]\left(\overrightarrow{x_0}+t_{i0}\left(\overrightarrow{z}-\overrightarrow{z_{0i}}\right)\right).$$

I.e.,

$$\left\|\left\|g_{i\overrightarrow{z}}^{(n+1)}\right\|\right\|_{\infty,[0,1]}\leq\left[\left(\sum_{\lambda=1}^k\left|z_\lambda-x_{0\lambda}\right|\left\|\left\|\frac{\partial}{\partial z_\lambda}\right\|\right\|_{\infty}\right)^{n+1}f_i\right], \quad (26)$$

$i=1,...,r.$

Hence by (26) we get

$$\left\|R_{in}\left(\overrightarrow{z},0\right)\right\|\leq\frac{\left[\left(\sum_{\lambda=1}^k\left|z_\lambda-x_{0\lambda}\right|\left\|\left\|\frac{\partial}{\partial z_\lambda}\right\|\right\|_{\infty}\right)^{n+1}f_i\right]}{(n+1)!}\leq$$

$$\frac{D_{n+1}\left(f_i\right)}{(n+1)!}\left(\sum_{\lambda=1}^k\left|z_\lambda-x_{0\lambda}\right|\right)^{n+1}=\frac{D_{n+1}\left(f_i\right)}{(n+1)!}\left\|\overrightarrow{z}-\overrightarrow{x_0}\right\|_{l_1}^{n+1}, \quad (27)$$

$i=1,...,r.$

Therefore it holds

$$\left\| R_{in}\left(\overrightarrow{z},0\right)\right\| \leq \frac{\max\limits_{i\in\{1,...,r\}} D_{n+1}\left(f_i\right)}{(n+1)!}\left\|\overrightarrow{z}-\overrightarrow{x_0}\right\|_{l_1}^{n+1}, \tag{28}$$

for $i=1,...,r$.

By (22) we get that

$$\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{z}\right) - \left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{x_0}\right) = \left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) R_{in}\left(\overrightarrow{z},0\right), \tag{29}$$

for all $i=1,...,r$.

Hence

$$\sum_{i=1}^{r}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{z}\right) - \sum_{i=1}^{r}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{x_0}\right)$$

$$= \sum_{i=1}^{r}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) R_{in}\left(\overrightarrow{z},0\right). \tag{30}$$

Therefore we find

$$E\left(f_1,...,f_r\right)\left(x_0\right) :=$$

$$\sum_{i=1}^{r}\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{z}\right)d\overrightarrow{z} - \sum_{i=1}^{r}\left(\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right)d\overrightarrow{z}\right) f_i\left(\overrightarrow{x_0}\right) =$$

$$\sum_{i=1}^{r}\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) R_{in}\left(\overrightarrow{z},0\right)d\overrightarrow{z}. \tag{31}$$

Consequently, we have that

$$\left\| E\left(f_1,...,f_r\right)\left(x_0\right)\right\| =$$

$$\left\|\sum_{i=1}^{r}\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{z}\right)d\overrightarrow{z} - \sum_{i=1}^{r}\left(\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right)d\overrightarrow{z}\right) f_i\left(\overrightarrow{x_0}\right)\right\| =$$

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$$\left\| \sum_{i=1}^{r} \int_{Q} \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_{\rho}\left(\overrightarrow{z}\right) \right) R_{in}\left(\overrightarrow{z}, 0\right) d\overrightarrow{z} \right\| \leq \tag{32}$$

$$\sum_{i=1}^{r} \left\| \int_{Q} \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_{\rho}\left(\overrightarrow{z}\right) \right) R_{in}\left(\overrightarrow{z}, 0\right) d\overrightarrow{z} \right\| \leq$$

$$\sum_{i=1}^{r} \left(\int_{Q} \left\| \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_{\rho}\left(\overrightarrow{z}\right) \right) R_{in}\left(\overrightarrow{z}, 0\right) \right\| d\overrightarrow{z} \right) \overset{(6)}{\leq}$$

$$\sum_{i=1}^{r} \left(\int_{Q} \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| f_{\rho}\left(\overrightarrow{z}\right) \right\| \right) \left\| R_{in}\left(\overrightarrow{z}, 0\right) \right\| d\overrightarrow{z} \right) \overset{(28)}{\leq} \tag{33}$$

$$\frac{\max\limits_{i \in \{1,...,r\}} D_{n+1}\left(f_i\right)}{(n+1)!} \sum_{i=1}^{r} \left(\int_{Q} \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| f_{\rho}\left(\overrightarrow{z}\right) \right\| \right) \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1} d\overrightarrow{z} \right).$$

So far we have proved

$$\left\| E\left(f_1, ..., f_r\right)\left(x_0\right) \right\| \leq$$

$$\frac{\max\limits_{i \in \{1,...,r\}} D_{n+1}\left(f_i\right)}{(n+1)!} \sum_{i=1}^{r} \left(\int_{Q} \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| f_{\rho}\left(\overrightarrow{z}\right) \right\| \right) \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1} d\overrightarrow{z} \right) =: (\xi). \tag{34}$$

Furthermore it holds

$$(\xi) \leq \frac{\max\limits_{i \in \{1,...,r\}} D_{n+1}\left(f_i\right)}{(n+1)!} \left(\int_{Q} \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1} d\overrightarrow{z} \right) \left[\sum_{i=1}^{r} \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| \left\| f_{\rho} \right\| \right\|_{\infty, Q} \right) \right], \tag{35}$$

and

$$(\xi) \leq \frac{\max\limits_{i \in \{1,...,r\}} D_{n+1}\left(f_i\right)}{(n+1)!} \left\| \left\| \cdot - \overrightarrow{x_0} \right\|_{l_1}^{n+1} \right\|_{\infty, Q} \left[\sum_{i=1}^{r} \left\| \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| f_{\rho} \right\| \right) \right\|_{L_1(Q,A)} \right], \tag{36}$$

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and finally

$$(\xi) \leq \frac{\max\limits_{i\in\{1,\dots,r\}} D_{n+1}(f_i)}{(n+1)!}\left[\sum_{i=1}^{r}\left[\left\|\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r}\|f_\rho\|\right)\right\|_{L_q(Q,A)}\right]\right]\left\|\|\cdot-\overrightarrow{x_0}\|_{l_1}^{n+1}\right\|_{L_p(Q,A)}, \tag{37}$$

proving (18), (19). ∎

We give

Corollary 9 *(to Theorem 8) All as in Theorem 8, with $f_1 = \dots = f_r = f$, $r \in \mathbb{N}$. Then*

$$\left\|\int_Q f^r(\overrightarrow{z})\,d\overrightarrow{z} - \left(\int_Q f^{r-1}(\overrightarrow{z})\,d\overrightarrow{z}\right) f(\overrightarrow{x_0})\right\| \leq$$

$$\frac{D_{n+1}(f)}{(n+1)!}\left(\int_Q \|f(\overrightarrow{z})\|^{r-1}\,\|\overrightarrow{z}-\overrightarrow{x_0}\|_{l_1}^{n+1}\,d\overrightarrow{z}\right) \leq \tag{38}$$

$$\frac{D_{n+1}(f)}{(n+1)!}\min\left\{\left(\int_Q \|\overrightarrow{z}-\overrightarrow{x_0}\|_{l_1}^{n+1}\,d\overrightarrow{z}\right)\left(\|\|f\|\|_{\infty,Q}\right)^{r-1},\right.$$

$$\left.\left\|\|\cdot-\overrightarrow{x_0}\|_{l_1}^{n+1}\right\|_{\infty,Q}\left\|\|f\|^{r-1}\right\|_{L_1(Q,A)}, \left\|\|\cdot-\overrightarrow{x_0}\|_{l_1}^{n+1}\right\|_{L_p(Q,A)}\left\|\|f\|^{r-1}\right\|_{L_q(Q,A)}\right\}. \tag{39}$$

We also give

Corollary 10 *(to Theorem 8) All as in Theorem 8, with $(A, \|\cdot\|)$ being a commutative Banach algebra. Then*

$$\left\|r\int_Q\left(\prod_{\rho=1}^{r} f_\rho(\overrightarrow{z})\right)d\overrightarrow{z} - \sum_{i=1}^{r}\left(\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho(\overrightarrow{z})\right)d\overrightarrow{z}\right) f_i(\overrightarrow{x_0})\right\| \leq$$

$$\textit{Right hand side of (18)} \;\leq\; \textit{Right hand side of (19)}. \tag{40}$$

We make

Remark 11 *Of great interest are applications of Theorem 8 when $Q = \prod_{\lambda=1}^{k}[a_\lambda, b_\lambda]$, where $[a_\lambda, b_\lambda] \subset \mathbb{R}$, $\lambda = 1, \dots, k$.*

We observe that by the multinomial theorem we get:

$$\int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]}\left(\sum_{\lambda=1}^{k}|z_\lambda - x_{0\lambda}|\right)^{n+1} dz_1\dots dz_k = \sum_{\rho_1+\rho_2+\dots\rho_k=n+1}\frac{(n+1)!}{\rho_1!\rho_2!\dots\rho_k!}$$

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$$\int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]} |z_1 - x_{01}|^{\rho_1} |z_2 - x_{02}|^{\rho_2} \dots |z_k - x_{0k}|^{\rho_k} dz_1 \dots dz_k = \tag{41}$$

$$\sum_{\rho_1+\rho_2+\dots\rho_k=n+1} \frac{(n+1)!}{\rho_1!\rho_2!\dots\rho_k!} \prod_{\lambda=1}^{k} \left(\int_{a_\lambda}^{b_\lambda} |z_\lambda - x_{0\lambda}|^{\rho_\lambda} dz_\lambda \right) =$$

$$\sum_{\sum_{\lambda=1}^{k}\rho_\lambda=n+1} \frac{(n+1)!}{\prod_{\lambda=1}^{k}\rho_\lambda!} \prod_{\lambda=1}^{k} \left(\int_{a_\lambda}^{x_{0\lambda}} (x_{0\lambda} - z_\lambda)^{\rho_\lambda} dz_\lambda + \int_{x_{0\lambda}}^{b_\lambda} (z_\lambda - x_{0\lambda})^{\rho_\lambda} dz_\lambda \right) =$$

$$\sum_{\sum_{\lambda=1}^{k}\rho_\lambda=n+1} \frac{(n+1)!}{\prod_{\lambda=1}^{k}\rho_\lambda!} \prod_{\lambda=1}^{k} \left(\frac{(x_{0\lambda} - a_\lambda)^{\rho_\lambda+1} + (b_\lambda - x_{0\lambda})^{\rho_\lambda+1}}{\rho_\lambda+1} \right). \tag{42}$$

We have found that

$$\int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]} \|\overrightarrow{z} - \overrightarrow{x_0}\|_{l_1}^{n+1} d\overrightarrow{z} = \tag{43}$$

$$\sum_{\sum_{\lambda=1}^{k}\rho_\lambda=n+1} \frac{(n+1)!}{\prod_{\lambda=1}^{k}\rho_\lambda!} \prod_{\lambda=1}^{k} \left(\frac{(b_\lambda - x_{0\lambda})^{\rho_\lambda+1} + (x_{0\lambda} - a_\lambda)^{\rho_\lambda+1}}{\rho_\lambda+1} \right).$$

Based on (18), (19) and (43) we conclude:

Theorem 12 *Let $(A, \|\cdot\|)$ a Banach algebra and $f_i \in C^{n+1}\left(\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda], A\right)$, $i = 1,\dots,r$; $r \in \mathbb{N}$, $n \in \mathbb{Z}_+$, and fixed $\overrightarrow{x_0} \in \prod_{\lambda=1}^{k}[a_\lambda,b_\lambda] \subset \mathbb{R}^k$, $k \geq 2$. Here all vector partial derivatives $f_{i\alpha} := \frac{\partial^\alpha f_i}{\partial z^\alpha}$, where $\alpha = (\alpha_1,\dots,\alpha_k)$, $\alpha_\lambda \in \mathbb{Z}^+$, $\lambda = 1,\dots,k$, $|\alpha| = \sum_{\lambda=1}^{k}\alpha_\lambda = j$, $j = 1,\dots,n$, fulfill $f_{i\alpha}(\overrightarrow{x_0}) = 0$, $i = 1,\dots,r$.*

Denote

$$D_{n+1}(f_i) := \max_{\alpha:|\alpha|=n+1} \| \|f_{i\alpha}\| \|_{\infty,\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]}, \tag{44}$$

$i = 1,\dots,r$.

Then

$$\left\| \sum_{i=1}^{r} \int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]} \left(\prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} f_\rho(\overrightarrow{z}) \right) f_i(\overrightarrow{z}) d\overrightarrow{z} - \right.$$

$$\left. \sum_{i=1}^{r} \left(\int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]} \left(\prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} f_\rho(\overrightarrow{z}) \right) d\overrightarrow{z} \right) f_i(\overrightarrow{x_0}) \right\| \leq \tag{45}$$

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$$\left(\max_{i\in\{1,\ldots,r\}} D_{n+1}\left(f_i\right)\right)\left[\sum_{i=1}^{r}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} \left\|\left\|\left\|f_\rho\right\|\right\|\right\|_{\infty,\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]}\right)\right]$$

$$\left[\sum_{\sum_{\lambda=1}^{k}\rho_\lambda=n+1} \frac{1}{\prod_{\lambda=1}^{k}\rho_\lambda!\prod_{\lambda=1}^{k}\left(\rho_\lambda+1\right)} \prod_{\lambda=1}^{k}\left(\left(b_\lambda-x_{0\lambda}\right)^{\rho_\lambda+1}+\left(x_{0\lambda}-a_\lambda\right)^{\rho_\lambda+1}\right)\right].$$

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Gap Formula for the Mexican hat wavelet transform

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Abstract

In this paper, we study the Mexican hat wavelet formulated from the Gaussian function. The Mexican hat wavelet transform (MHWT) is defined using this basic wavelet. A standard method is introduced to obtain the gap formula for the MHWT. Further, an example for the gap formula is also presented.

Key words: Fourier transform; Wavelet transform; Schwartz distributions; Tempered Boehmians

Mathematics Subject Classification(2010): 44A15; 46F12; 54B15; 46F99

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1 Introduction

By utilizing the theory of distributional as well as classical Fourier and Hilbert transforms, the theory of wavelet transform in L^p-spaces $(1 \leq p \leq \infty)$ is formulated. The wavelet transform has been rising as a major mathematical tool for the past two decades and its contribution to signal analysis is significant. The major reason for this is the representation of functions in a time-frequency plane is possible with wavelet transform. Hence, the wavelet transform can be treated as an operator which localizes time and frequency. Moreover, one can regulate wavelets within a fixed time period to acquire varied frequency components that are useful in enhancing the study of signals having localized impulses and oscillations. Based on the idea of wavelets as a family of functions, the mother wavelet $\psi_{b,a}(t)$ is defined by dilating and translating the function $\psi \in L^2(\mathbb{R})$ and is given by

$$\psi_{b,a}(u) = (\sqrt{a})^{-1}\psi\left(\frac{u-b}{a}\right), \quad b, u \in \mathbb{R}, a \in \mathbb{R}_+ = (0, \infty), \tag{1.1}$$

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where a is the dilation, which calculates the level of compression, and b is called shifting parameter, which works out the wavelet's time location. If $|a|< 1$, then (1.1) is the compressed version of the mother wavelet and represents higher frequencies.
For a square integrable function f, the wavelet transform with respect to $\psi_{b,a}$ is defined by [5],

$$W(b,a) = \int_{-\infty}^{\infty} f(u)\overline{\psi_{b,a}}(u)du \text{ for } a \in T_+ \text{ and } u, b \in \mathbb{R}. \tag{1.2}$$

The inversion formula for (1.2) is given as follows:

$$f(x) = \frac{2}{C_\psi}\int_0^\infty \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{a}} W(b,a)\psi\left(\frac{x-b}{a}\right) db\right] \frac{da}{a^2}, \quad x \in \mathbb{R} \tag{1.3}$$

where

$$\frac{1}{2}C_\psi = \int_0^\infty \frac{|\hat{\psi}(u)|^2}{|u|}du = \int_0^\infty \frac{|\hat{\psi}(-u)|^2}{|u|}du < \infty \qquad [1, \text{p. } 64].$$

Recently among very many authors, the researches carried out by R. S. Pathak *et al.* [4-10] have investigated the theory of wavelet transform to distributions and ultradistribution spaces. Singh *et al.* have extended the theory for distributional wavelet and mexican hat wavelet transform [11-14]. Further, inversion formulae for the same are established in the sense of distributions and ultradistributions.

Mexican hat wavelet that is formulated by taking the second derivative of Gaussian function is defined by

$$\psi(u) = exp\left(\frac{-u^2}{2}\right)\ (1-u^2) = -\frac{d^2}{du^2}\ exp\left(\frac{-u^2}{2}\right). \tag{1.4}$$

Therefore,

$$\psi_{b,a}(u) = -a^{3/2} D_u^2 exp\left(-\frac{(b-u)^2}{2a^2}\right), \qquad \left(D_u = \frac{d}{du}\right). \tag{1.5}$$

Thus from (1.2), we have

$$W(b,a) = -a^{3/2}\int_{-\infty}^{\infty} f(t)\ D_t^2\ exp\left(-\frac{(b-t)^2}{2a^2}\right) dt, \qquad a > 0. \tag{1.6}$$

Then, under certain conditions on f, we have

$$W(b,a) = -a^{3/2}\int_{-\infty}^{\infty} f^{(2)}(t)\ exp\left(-\frac{(b-t)^2}{2a^2}\right) dt, \qquad a > 0. \tag{1.7}$$

From the above two equations we can consider the MHWT as the Weierstrass transform of $\left(\frac{d}{du}\right)^2 f(u)$. This relation can further be utilized to explore various

properties of $W(b,a)$. Also, as Weierstrass transform is defined for complex values of b, therefore, the definition of the MHWT can be extended for b being complex, whenever required.

Now for $a \in (0,\infty)$ and $b \in \mathbb{C}$, we define

$$k(b,a) = \frac{1}{\sqrt{2\pi a}} exp\left(\frac{-b^2}{2a}\right). \tag{1.8}$$

Clearly,

$$D_u^2 k(b-u,a^2) = \frac{1}{\sqrt{2\pi a}} D_u^2\left(exp\left(\frac{-(b-u)^2}{2a^2}\right)\right). \tag{1.9}$$

Hence the Mexican hat wavelet transform of a function $f(t)$ is given by [7]

$$W(b,a) = a^{3/2}\int_{-\infty}^{\infty} f^{(2)}(u) exp\left(\frac{-(b-u)^2}{2a^2}\right) du. \tag{1.10}$$

2 Gap formula for Mexican hat wavelet transform

The gap formula which is also known as the jump operator provides a unified approach to obtain a relation between the determining function at a given point in terms of the transform. Here, it acts as an operator which gives $f^{(2)}(b+) - f^{(2)}(b-)$ in terms of $W(b,a)$ where $W(b,a)$ and $f^{(2)}(b)$ are related by (1.10). Such representations have been obtained for various integral transform like Laplace transform, Stieltjes transform, Weierstrass transform, and many more [2, 15, 16]. In the next theorem, we present Gap formula for the Mexican hat wavelet transform.

Theorem 2.1. *Let $f^{(2)}(y) \in L_1(m,n)$ for any finite interval such that the integral (1.10) relating $W(b,a)$ to $f^{(2)}(y)$ converges for $m < b < n$. Also, there exists numbers $f^{(2)}(b\pm 0)$ satisfying*

$$\int_0^h [f^{(2)}(b\pm u) - f^{(2)}(b\pm 0)]du = o(h), \quad h \to 0.$$

Then for d satisfying $m < d < n$ we have for $-\infty < b < \infty$,

$$\lim_{a^2\to 1-} -i(1-a^2)^{3/2} a\int_{d-i\infty}^{d+i\infty} (s-b)\exp\left(\frac{(s-b)^2}{2a^2}\right) W(s,1)ds = f^{(2)}(b+0) - f^{(2)}(b-0).$$

Proof. Let $\alpha(u) = \int_0^u f^{(2)}(v)dv,\ \forall d \in (m,n)$. Also, let $\alpha(u)$ be locally bounded variation, such that

$$|\alpha(u)| = \begin{cases} M\exp\left(\dfrac{(u-\eta)^2}{2}\right), & u > x, \\ M\exp\left(\dfrac{(u-\xi)^2}{2}\right), & u < x. \end{cases} \tag{2.1}$$

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Then the MHWT of $f(v)$ is defined by

$$W(b,1)=\int_{-\infty}^{\infty}k(b-u,1)f^{(2)}(v)dv. \tag{2.2}$$

Now, using integration by parts on (2.2), we get

$$W(b,1)=\int_{-\infty}^{\infty}k_1(b-u,1)\alpha(u)du, \tag{2.3}$$

where

$$k_1(b-u,1)=\frac{\partial}{\partial b}k(b-u,1).$$

Consider

$$\begin{aligned}
I &= -i(1-a^2)^{13/2}\int_{d-i\infty}^{d+i\infty}(s-b)\exp\left(\frac{(s-b)^2}{2a^2}\right)W(s,1)ds\\
&= -i(1-a^2)^{3/2}\int_{d-i\infty}^{d+i\infty}(s-b)\exp\left(\frac{(s-b)^2}{2a^2}\right)\int_{-\infty}^{\infty}k_1(s-u,1)\alpha(u)du\\
&= -i(1-a^2)^{3/2}\sqrt{2\pi}a\int_{-\infty}^{\infty}\alpha(u)du\int_{d-i\infty}^{d+i\infty}\frac{(s-b)}{\sqrt{2\pi}a}\exp\left(\frac{(s-b)^2}{2a^2}\right)k_1(s-u,1)ds.
\end{aligned}$$

Let us consider

$$\begin{aligned}
J &= \frac{-i}{\sqrt{2\pi}a}\int_{d-i\infty}^{d+i\infty}(s-b)\exp\left(\frac{(s-b)^2}{2a^2}\right)k_1(s-u,1)ds\\
&= \frac{1}{\sqrt{2\pi}a}\int_{-\infty}^{\infty}(d+iy-b)\exp\left(\frac{(d+iy-b)^2}{2a^2}\right)k_1(d+iy-u,1)dy, \quad (s=d+iy)\\
&= \frac{1}{\sqrt{2\pi}a}\int_{-\infty}^{\infty}i(y-i(d-b))\exp\left(\frac{-(y-i(d-b))^2}{2a^2}\right)k_1(iy+d-u,1)dy\\
&= \int_{-\infty}^{\infty}k(d+iy-b,a^2)k_2(d+iy-u,1)dy,
\end{aligned}$$

where

$$k_2(s-u,1)=\frac{\partial^2 k(s-u,1)}{\partial s^2}=(s-u)k_1(s-u,1).$$

By [7, Theorem 2.1], we have

$$\begin{aligned}
J &= \int_{-\infty}^{\infty}k(d+iy-b,a^2)k_2(d+iy-u,1)dy \qquad (2.4)\\
&= k_2(d+iy-u-d-iy+b,1-a^2)\\
&= k_2(b-u,1-a^2).
\end{aligned}$$

Hence, we obtain $J=k_2(b-u,1-a^2)$, by combining (2.4) with Corollary 2.2 of [3], where $f^{(2)}(b)=k_2(b-u,1-a^2)$. Further, breaking the integral I into

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4 parts, corresponding to the intervals $(-\infty, b-\delta)$, $(b-\delta, b)$, $(b, b+\delta)$ and $(b+\delta, \infty)$, we have

$$\begin{aligned} I &= (1-a^2)^{3/2}(2\pi)^{1/2}a\left\{\int_{-\infty}^{b-\delta}+\int_{b-\delta}^{b}+\int_{b}^{b+\delta}+\int_{b+\delta}^{\infty}\right\}\alpha(u)(u)k_2(b-u,1-a^2)du \\ &= I_1(a)+I_2(a)+I_3(a)+I_4(a). \end{aligned}$$

For $I_2(a)$, we can choose a $\delta > 0$ so that $|f^{(2)}(u)-f^{(2)}(b-)|< \epsilon$ for $b-\delta < u < b$ and therefore,

$$\begin{aligned} |I_2(a)+f^{(2)}(b-)| &= \left|\int_{b-\delta}^{b} k_1(b-u,1-a^2)[f^{(2)}(u)-f^{(2)}(b-)]du\right| + o(1) \\ &= \left|\int_{b-\delta}^{b} k_2(b-u,1-a^2)\beta(u)du\right| + o(1) \\ &\leq \epsilon\int_{b-\delta}^{b} k_2(b-u,1-a^2)|s-u|du + o(1) \\ &\leq \epsilon M + o(1) \quad as\ a^2 \to 1-. \end{aligned}$$

Similarly $|I_3(a)-f^{(2)}(b+)|\leq \epsilon M + o(1)$.

For ϵ being arbitrary, we have $I_2(a) \approx -f^{(2)}(b-)$ and $I_3(a) \approx f^{(2)}(b+)$.

For $I_1(a)$ and $I_4(a)$ by Lemma 2.1c of [3], for some ξ and η such that $m < \xi < \eta < n$, at $a = 1$

$$\begin{aligned} f^{(2)}(u) &= o\left[exp\left(\frac{(u-\eta)^2}{2}\right)\right], \quad u \to \infty, \\ f^{(2)}(u) &= o\left[exp\left(\frac{(u-\xi)^2}{2}\right)\right], \quad u \to \infty. \end{aligned}$$

Therefore,

$$\begin{aligned} |I_1(a)| &= \lim_{a^2\to 1-}\left|(2\pi)^{1/2}(1-a^2)^{3/2}\int_{-\infty}^{b-\delta} k_1(b-u,1-a^2)f^{(2)}(u)du\right| \\ &\leq \lim_{a^2\to 1-}(1-a^2)^{-3/2}\int_{-\infty}^{b-\delta}\exp\left(\frac{-(b-u)^2}{2(1-a^2)}\right)|f^{(2)}(u)|du \\ &\leq \lim_{a^2\to 1-}M(1-a^2)^{-3/2}\int_{-\infty}^{b-\delta}\exp\left(\frac{-(b-u)^2}{2(1-a^2)}\right)\exp\left(\frac{-(u-\xi)^2}{2}\right)du \\ &= \mathrm{o}(1). \end{aligned}$$

Hence, $I_1(a) = \mathrm{o}(1)$ and similarly $I_4(a) = \mathrm{o}(1)$ as $a^2 \to 1-$, which concludes the proof of the theorem.

□

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Example 2.2. As a simple example take the MHWT at $a = 1$,

$$\begin{aligned} W(s,1) &= \int_{-\infty}^{\infty} k_1(s-u,1)\alpha(u)du \\ &= \exp\left(\frac{-s^2}{2}\right), \end{aligned} \tag{2.5}$$

where

$$\alpha(u) = \int_0^u f^{(2)}(v)dv = \begin{cases} 0 & u < 0 \\ 1 & u > 0. \end{cases}$$

Since the integral (1.10) converges always, therefore by Theorem 2.1, we have

$$\begin{aligned} &= \lim_{a^2\to 1-} -i(1-a^2)^{3/2} \int_{-\infty}^{\infty} (s-b)\exp\left(\frac{(s-b)^2}{2a^2}\right) W(s,1)ds \\ &= \lim_{a^2\to 1-} -i(1-a^2)^{3/2} \int_{-\infty}^{\infty} (s-b)\exp\left(\frac{(s-b)^2}{2a^2}\right)\exp\left(\frac{-s^2}{2}\right) ds \\ &= \lim_{a^2\to 1-} \frac{i(1-a^2)^{3/2}\sqrt{2\pi}a^4}{(a^2-1)^{3/2}} \exp\left(\frac{-b^2}{2(1-a^2)}\right) \\ &= \begin{cases} 1 & b = 0, \\ 0 & otherwise. \end{cases} \end{aligned} \tag{2.6}$$

Conclusions

In this article, we studied the conditions needed to obtain a relation between the determining function at a point of discontinuity with its MHWT. As the Gaussian function derives the Mexican hat wavelet, therefore it satisfies the Gaussian decays in both frequency and space. Further, as the MHWT has localization in both space and frequency, it has a strong appeal to applications in space-frequency analysis, mixed boundary value problems, approximation theory, mathematical modeling, other digital modulation.

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Modelling the fear effect in prey predator ecosystem incorporating prey patches

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Abstract

In an ecosystem, the balance of prey-predator system is greatly influenced by the availability of prey and the fear imposed on it's population. In this paper, it is proposed that a prey-predator model in which prey is assumed to be able to detect the presence of predator and to counteract it by forming patches and incorporating the cost of fear into prey reproduction. **Equilibrium points are calculated and analysis of the local and global asymptotic behaviors of the system are done. Hopf-bifurcation is seen in case of adequate availability of prey. The system stabilizes in presence of high levels of fear.** Availability of prey act as a crucial role to change the dynamics of the system. **Numerical simulations showcases the relationship between prey patches and other related parameters like level of fear, conversion rate of predator and availability of prey. These simulations reveal the impact of fear on the prey-predator system and also justify the theoretical findings.** In the end, the bifurcation scenarios are derived when two different parameters switch together at a same time. Numerical simulations are justified the theoretical findings.

Keywords: Fear; Patches; Hunting Stability; Bifurcation.

1 Introduction

The survey of prey-predator dynamics is one of the blooming topics of ecosystem in last few decades. Predation process perform an indispensable part to maintain ecological balance. In real field application, the predator do not capture all the prey population due to refuge property of prey [1, 2]. In biomathematics, the research of prey refuge is one of the hot spot area. As a result, many researchers focus in this aspect [3, 4, 5]. Some experimental finding confirm that fear effect

on predator may alter the behavior of prey [6, 7, 8]. Some theoretical studies have revealed that growth rate of prey need to improve through implementation of fear effects [9, 10, 11]. Recently, the authors in [12] studied the hunting cooperation and the fear factor among prey in a Leslie-Gower model. This study revealed that fear factor is more effective than hunting cooperation to stabilize the system. Also, the scientists in [13] proposed a Beddington-DeAngelis functional response of predator-prey model and investigated the impact of anti-predator activity on whole system. They noted that the system may exhibits multiple Hopf-bifurcation. The researchers in [14] investigated that chaotic system turned into stable system in presence of cost of fear in three species model. But very few numbers of researchers explored the combine effects of hunting cooperation and anti-predator activity in predator-prey system. In recent past, the authors in [15] studied the combine effects of hunting cooperation and fear factor in prey-predator system and observed that strong demographic Allee phenomenon. **Recently, the authors in [16] studies the influence of harvesting and allee effects in disease induced prey-predator system and reveals that allee effect and harvesting can be a handy technique for controlling the spread of disease. Fractional order mathematical models are a new research field in non-linear dynamics [17, 18]. The authors in [19] apply the homotopy analysis transform technique in prey-predator model to evaluate approximate solution which converges to the exact solution of time-fractional nonlinear subject to initial conditions.**

Anti-grazing strategy is a vital part in prey-predator system to protect prey from predator. In marine system, size of phytoplankton are very small compare to the predatory enemies but they can survive from consumes by using anti-grazing strategies like morphology [20] formation of colonies [21] which resist the grazing pressure by higher trophic organisms. Toxin ejected by phytoplankton is one of another anti-grazing strategies to protect from zooplankton [22]. The author in [23] studied the formulation of patches for defense mechanism and discussed the ability of releasing toxin chemicals. Thus, paired mechanism over with patching and poison release outcomes will act a crucial role for the coexistence species. Some experimental researches noted that the patch size depend on organism density and also proportional with it [24]. In real field, phytoplankton are allowed to form spherical patches or colonies and release toxin chemicals [25].

Motivated by the above theoretical and experimental literatures, the dynamics of such system in which hunting by predator and fear of prey is studied. The aim of the present study is to investigate the impact of hunting, fear effect and toxin effect due to formulation of patches. As per my knowledge, the combine effect of three above factors has not to explore yet. **The main target in present manuscript is to investigate the subsequent biological topics:**

• How does availability of prey density influence on the dynamics of prey-predator system.

• Can fear factor among prey influence to stabilize the prey-predator system.

• How does patches influence the prey-predator dynamics.

It is considered that, birth rate of prey population is reduced due to fear of hunting by predator. In the next section, proposed model is developed with incorporate prey patches. Section 2 represents the construction of mathematical model based on some assumptions. Basic properties such as boundedness is discussed in Section 3. Analytical results based on the model and **global stability** are discussed in Section 4. Section 5 represents the local bifurcation such as Hopf and transcritical-bifurcation analysis. Numerical simulations and discussion are illustrated in Section 6 & 7. Finally, the paper summarize with a brief conclusion.

2 Basic assumptions and model formulation

Let us consider the assumption to construct the following mathematical model: Let $x(t)$ and $y(t)$ be the density of prey and predator population at time $t > 0$ respectively. Here r and r_1 be the intrinsic growth rate and the intra-species competition rate of prey. c **and** e **represent the predation rate and conversion rate of predator.** Here $(1-k_1)$ terms represents the amount of availability of prey for predation by the predator where, $k_1 \in (0, 1]$. It is assumed that predation term is the Holling-II functional form. According to literature review, a fraction part k_1 of prey aggregate to form N patches. Therefore, each patches represent as $\frac{1}{N}k_1x$. It is assume that the three dimensional patch is roughly spherical in ocean. Therefore, the radius of patch is proportional to $[\frac{1}{N}k_1x]^{1/3}$. As a result the surface of patch is proportional to $[\frac{1}{N}k_1x]^{2/3} = \rho x^{2/3}$, where $\rho = [\frac{1}{N}k_1]^{2/3}$. **The effect of fear has a direct impact on prey reproduction [26, 27, 28]**. In presence of predator, intrinsic growth of prey becomes a function of the predator density like $F(y; K) = \frac{r}{1+Ky}$ in which K is defined as level of fear of the prey according to anti-predator response. **This above function follows some conditions**:

(i) $F(y; 0) = r$**:** in the absence of fear effect, the prey reproduction rate remain unaltered.

(ii) $F(0; K) = r$**:** in the absence of predator, **the prey reproduction rate remain unaltered**.

(iii) $\lim_{K\to\infty} F(y; K) = 0$**: extremely fearful prey fails to reproduce.**

(iv) $\lim_{y\to\infty} F(y; K) = 0$**: at a extremely higher predator density, prey fails to reproduce.**

(v) $\frac{\partial F(y;K)}{\partial K} < 0$**:** the prey reproduction rate low with high amount of fear effect.

(vi) $\frac{\partial F(y;K)}{\partial y} < 0$**:** the prey reproduction rate low with high amount of predator density.

$$\begin{aligned} \frac{dx}{dt} &= \frac{rx}{1+Ky} - r_1x^2 - \frac{c(1-k_1)xy}{1+a(1-k_1)x} \equiv G_1(x,y) \\ \frac{dy}{dt} &= \frac{e(1-k_1)xy}{1+a(1-k_1)x} - dy - e\rho x^{2/3}y \equiv G_2(x,y). \end{aligned} \tag{1}$$

The system (1) will be analyzed with the following initial conditions,

$$x(0) \geq 0, \; y(0) \geq 0. \tag{2}$$

3 Mathematical preliminaries

Theorem 1. *All non negative solutions $(x(t), y(t)$ of the system (1) initiate in $R_+^2 - \{0,0\}$ are uniformly bounded.*

Proof. Let us choose a function $\Theta = x + y$.
Therefore,

$$\frac{d\Theta}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = \frac{rx}{1+Ky} - r_1x^2 - \frac{c(1-k_1)xy}{1+a(1-k_1)x} + \frac{e(1-k_1)xy}{1+a(1-k_1)x} - dy - e\rho x^{2/3}y.$$

Let us consider a positive constant ζ such that $\zeta \leq d$. Therefore,

$$\frac{d\Theta}{dt} + \zeta\Theta \leq r_0x - r_1x^2 + \zeta x - \frac{(1-k_1)(c-e)}{1+a(1-k_1)x} - y(d-\zeta) - e\rho x^{2/3}y$$

$$\leq (r_0+\zeta)x - r_1x^2 \leq \frac{(r_0+\zeta)^2}{4r_1}.$$

By choosing $\Gamma = \frac{(r_0+\zeta)^2}{4r_1}$, we obtain

$$0 \leq \Theta(x(t), y(t)) \leq \frac{\Gamma}{\zeta}(1 - e^{-\zeta t}) + \Theta(x(0), y(0))e^{-\zeta t},$$

which indicates that $0 \leq \Theta(x(t), y(t)) \leq \frac{\Gamma}{\zeta}$ as $t \to \infty$. Therefore, all non negatives solutions of the system (1) are originated from $R_+^2 - \{0,0\}$ will be restricted in the region $\nabla = \{(x,y) \in R_+^2 : x(t) + y(t) \leq \frac{\Gamma}{\zeta} + \varepsilon\}$.
In ecology, it means that the system act in a specified manner. Boundedness of the system implies that none of the two interacting species grow unexpectedly or exponentially for a long period of time. Clearly, as a result of limited resource, numbers of each species is surely bounded. □

From the ecological point of view, let us first consider the following region $R_+^2 = \{(x,y) : x \geq 0, y \geq 0\}$. Here, the function $G_1(x,y) = xf(x,y)$ and $G_2 = yg(x,y)$ of the system (1) are continuously differentiable and locally Lipschitz in $R_+^2 = \{(x,y) : x \geq 0, y \geq 0\}$. Therefore, Theorem A.4, page 423 in H. R. Thieme's book [29] implies that the solutions of the initial value problem with non-negative initial conditions exist on the interval $[0, S)$ and unique, where S is a sufficiently large number.

4 Equilibria: Existence and stability

All possible equilibria are catalogued below:
(**i**) The predator free equilibrium $E_1 = (\frac{r}{r_1}, 0)$.
(**ii**) The positive coexistence equilibrium $E^* = (x^*, y^*)$,
while x^* is ensured by solving $\{a(1-k_1)\}^3 e^3\rho^3 x^{*5} + 3\{a(1-k_1)\}^2 e^3\rho^3 x^{*4} + [3e^3\rho^3 a(1-k_1) - \{(1-k_1)(e-da)\}^3]x^{*3} + [e^3\rho^3 + 3\{(1-k_1)(e-da)\}^2 d]x^{*2} - 3\{(1-k_1)(e-da)\}d^2x^* + d^3 = 0$.
Also, y^* is ensured by solving $cK(1-k_1)y^2 + [c(1-k_1) + r_1x^*(1+a(1-k_1)x^*)K]y^* - (1+a(1-k_1)x^*)(r - r_1x^*) = 0$.

Thus the condition for the existence of the interior equilibrium point $E^*(x^*,\ y^*)$ is given by, $x^* > 0,\ y^* > 0$.

Explicitly, general form of the Jacobian matrix at $\overline{E} = (\overline{x}, \overline{y})$ is defined as

$$\overline{J} = \begin{bmatrix} \frac{r}{(1+K\overline{y})} - 2r_1\overline{x} - \frac{c(1-k_1)\overline{y}}{(1+a(1-k_1)\overline{x})^2} & -\frac{rK\overline{x}}{(1+K\overline{y})^2} - \frac{c(1-k_1)\overline{x}}{1+a(1-k_1)\overline{x}} \\ \frac{e(1-k_1)\overline{y}}{(1+a(1-k_1)\overline{x})^2} - \frac{2}{3}e\rho\overline{y}\frac{1}{\overline{x}^{1/3}} & \frac{e(1-k_1)\overline{x}}{1+a(1-k_1)\overline{x}} - d - e\rho\overline{x}^{2/3} \end{bmatrix}. \qquad (3)$$

There exists a feasible predator free steady state E_1 of the system (1) which is unstable if $\frac{d}{e} + \rho\frac{r}{r_1}^{2/3} < \frac{(1-k_1)r}{a(1-k_1)r + r_1}$.
The Jacobian matrix at E^* can be written as
$$J^* = \begin{bmatrix} \frac{r}{(1+Ky^*)} - 2r_1x^* - \frac{c(1-k_1)y^*}{(1+a(1-k_1)x^*)^2} & -\frac{rKx^*}{(1+Ky^*)^2} - \frac{c(1-k_1)x^*}{1+a(1-k_1)x^*} \\ \frac{e(1-k_1)y^*}{(1+a(1-k_1)x^*)^2} - \frac{2}{3}e\rho\frac{y^*}{x^{*1/3}} & 0 \end{bmatrix}.$$
Thus the eigenvalues in this case are obtained as roots of the quadratic
$\lambda^2 - tr(J^*) + det(J^*) = 0$,
$tr(J^*) = \frac{r}{(1+Ky^*)} - 2r_1x^* - \frac{c(1-k_1)y^*}{(1+a(1-k_1)x^*)^2}$,
$det(J^*) = [\frac{rK}{(1+Ky^*)^2} + \frac{c(1-k_1)}{1+a(1-k_1)x^*}][\frac{e(1-k_1)}{(1+a(1-k_1)x^*)^2} - \frac{2}{3}e\rho\frac{1}{x^{*1/3}}]x^*y^*$.
Now $tr(J^*) < 0$ if $\frac{r}{(1+Ky^*)} < 2r_1x^* + \frac{c(1-k_1)y^*}{(1+a(1-k_1)x^*)^2}$ as well as $det(J^*) > 0$ if $\rho < \frac{27}{8}\frac{(1-k_1)^3x^*}{(1+a(1-k_1)x^*)^6}$.
Therefore, according Routh–Hurwitz criterion we can admit that E^* is locally asymptotically stable providing the above two conditions are fulfilled.

Theorem 2. ***If the non negative equilibrium*** E^* ***exists, then*** (x^*, y^*) ***is globally asymptotically stable in the*** $x - y$ ***plane if*** $r_1 > \frac{c(1-k_1)^2a}{1+a(1-k_1)x^*}$.

Proof. Let us consider a Lyapunov function about E^*
$V = x - x^* - x^*ln\frac{x}{x^*} + \frac{c}{e}(1+a(1-k_1)x^*)(y - y^* - y^*ln\frac{y}{y^*})$.
Differentiating V with respect to t of the system (1), we get
$\frac{dV}{dt} = (x-x^*)(\frac{r}{1+Ky} - r_1x - \frac{c(1-k_1)y}{1+a(1-k_1)x}) + \frac{c}{e}(1+a(1-k_1)x^*)(y-y^*)(\frac{e(1-k_1)xy}{1+a(1-k_1)x} - dy - e\rho x^{2/3}y)$
$=(x-x^*)\left(\frac{rK(y-y^*)}{(1+Ky)(1+Ky^*)} - r_1(x-x^*) + \frac{c(1-k_1)(y-y^*)}{1+a(1-k_1)x} + \frac{c(1-k_1)^2a(x-x^*)}{[1+a(1-k_1)x][1+a(1-k_1)x^*]}\right) + \frac{c}{e}(1+a(1-k_1)x^*)(y-y^*)\left[\frac{e(1-k_1)(x-x^*)}{(1+a(1-k_1)x)(1+a(1-k_1)x^*)} - e\rho(x^{\frac{2}{3}} - x^{*\frac{2}{3}})\right]$.

After some calculation and simplification we get
$\leq -\left[r_1 - \frac{c(1-k_1)^2 a}{1+a(1-k_1)x^*}\right](x-x^*)^2 - \frac{rK}{(1+Ky)}(x-x^*)(y-y^*)$.

Clearly, $\dot{V}$ is negative definite if $r_1 > \frac{c(1-k_1)^2 a}{1+a(1-k_1)x^*}$. Therefore by LaSalle's theorem [30] E^* is globally asymptotically stable in $x-y$ plane. □

5 Local bifurcation

5.1 Hopf-Bifurcation

Theorem 3. *The necessary and sufficient conditions for Hopf bifurcation of the system (1) around E^* at $k_1 = k_1^c$ are $[tr(J^*)]_{k_1=k_1^c} = 0$, $[det(J^*)]_{k_1=k_1^c} > 0$ and $\frac{d}{dk_1}[tr(J^*)]_{k_1=k_1^c} \neq 0$.*

Proof. The condition $[tr(J^*)]_{k_1=k_1^c} = 0$ gives $\frac{r}{(1+Ky^*)} - 2r_1x^* - \frac{c(1-k_1)y^*}{(1+a(1-k_1)x^*)^2} = 0$, in which $[tr(J^*)]_{k_1=k_1^c} = 0$.
Now $[det(J^*)]_{k_1=k_1^c} > 0$ which is equivalent to the characteristic equation $\lambda^2 + [det(J^*)]_{k_1=k_1^c} = 0$ whose roots are purely imaginary,
For $k_1 = k_1^c$, the characteristic can be written as

$$\chi^2 + \omega = 0, \tag{4}$$

where $\omega = [det(J^*)]_{k_1=k_1^c} > 0$. Therefore, the above equation has two roots of the form $\chi_1 = +i\sqrt{\omega}$ and $\chi_2 = -i\sqrt{\omega}$. Let at any neighbouring point k_1 of k_1^c, we can express the above roots in general form like $\chi_{1,2} = \theta_1(k_1) + \pm i\theta_2(k_1)$, where $\theta_1(k_1) = \frac{tr(J^*)}{2}$ and $\theta_2(k_1) = \sqrt{det(J^*) - \frac{tr(J^*)}{4}}$. Now it is to be verified the transversality condition $\frac{d}{dk_1}(Re(\chi_j(k_1)))_{k_1=k_1^c} \neq 0$ for $j = 1, 2$.
Substituting $\chi_1 = \theta_1(k_1) + i\theta_2(k_1)$ in (4) and calculate the derivative, we have

$$\begin{aligned} 2\theta_1(k_1)\theta_1'(k_1) - 2\theta_2(k_1)\theta_2'(k_1) + \omega' &= 0, \\ 2\theta_2(k_1)\theta_1'(k_1) + 2\theta_1(k_1)\theta_2'(k_1) &= 0. \end{aligned} \tag{5}$$

Solving (5), we get
$\frac{d}{dk_1}(Re(\chi_j(k_1)))_{k_1=k_1^c} = \frac{-2\theta_1\omega'}{2(\theta_1^2+\theta_2^2)} \neq 0$, i.e., $\frac{d}{dk_1}[tr(J^*)]_{k_1=k_1^c} \neq 0$, which satisfy the transversality condition. This implies that the system undergoes a Hopf-bifurcation at $k_1 = k_1^c$. □

5.2 Transcritical-bifurcation

Theorem 4. *System (1) undergoes a transcritical bifurcation when the system parameters satisfy the restriction $k_1 = k_1^{TC}$. Here, k_1 is seen as the bifurcation parameter.*

Proof. For $k_1 = k_1^{TC}$, the Jacobian matrix J_1 of the system (1) around E_1 has one zero eigenvalue. Let U_1 and V_1 be the eigenvectors of the matrix J_1 and $(J_1)^T$ corresponding to zero eigenvalue respectively. Therefore, we obtain $U_1 = \left(-(\frac{r}{r_1} + \frac{c(1-k_1)}{r_1+a(1-k_1)r}) \;\; 1\right)^T$ and $V_1 = (0 \;\; 1)^T$. We have $F_{k_1}(x, y) = \begin{pmatrix} 0 & -y \end{pmatrix}^T$, $F_{k_1}\left(E_1; k_1 = k_1^{TC}\right) = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$ and $(V_1)^T F_{k_1}\left(E_1; k_1 = k_1^{TC}\right) = 0$.
Also, $DF_{k_1}\left(E_1; k_1 = k_1^{TC}\right) U_1 = (0 \;\; -1)^T$.
Therefore, we obtain $(V_1)^T \left[DF_{k_1}\left(E_1; k_1 = k_1^{TC}\right)(U_1)\right] = -1$.
Further, $(V_1)^T D^2F\left(E_1; k_1 = k_1^{TC}\right)(U_1, U_1)$
$= -2e\left[\frac{r_1^2(1-k_1)}{(r_1+a(1-k_1)r)^2} - \frac{2e\rho}{3}(\frac{r_1}{r})^{1/3}\right]\left[\frac{r_1}{r} + \frac{e(1-k_1)}{r_1+a(1-k_1)r}\right] < 0.$
By applying Sotomayor's theorem [31] we can conclude that the system experiences a transcritical bifurcation at E_1 when k_1 crosses k_1^{TC}.

□

6 Numerical simulations

In order to visualize the analytical finding, we perform the numerical simulation over the set of parametric values **[32, 33, 34]**

$$\begin{aligned} r = 1.2,\ r_1 = 0.05, K = 0.1,\ k_1 = 0.7, \\ c = 0.45,\ e = 0.25,\ a = 0.3,\ d = 0.1,\ \rho = 0.15. \end{aligned} \tag{6}$$

It is noted that the system (1) shows stable dynamics around at $E^*(3.06, 5.74)$ (cf. Fig. 1(a)).

6.1 Effect of k_1

It is observed that when availability of prey species is high for predation, i.e., the low value of k_1, the dynamical system switches to unstable behavior (viz. $k_1 = 0.66$). But high level of fear can stabilize the system (1) (viz. $K = 0.2$). It is illustrated in Fig. 1(b). **Thus, the fear effect can prevent the occurrence of limit cycle oscillation and increase the stability of the system.** Fig. 2(a-b) depicts various steady state behavior of prey and predator for the parameter k_1. Here, it is noted that a Hopf point are situated (H) at $k_1 = 0.673026$ with eigenvalue $\pm 0.284862i$ and one Limit point (LP) and a Branch point (BP) coincide at $k_1 = 0.864180$ with eigenvalue $(0. - 1.2)$. **Branch point (BP) indicates that at that particular point, predator goes to extension and the transcritical bifurcation occurs. The Limit point (LP) is a collision and disappearance of two equilibria in the dynamical system. The system switches from stable to unstable or unstable to stable behavior after crossing the Hopf point(H).** It is observed that the first Lyapunov coefficient being $-2.654148e^{-03}$ at Hopf point (H) which confirm that a family of stable limit cycle generate from H (viz. Fig. 3(a)). **It is clearly indicates that increasing the amount of prey refuge**

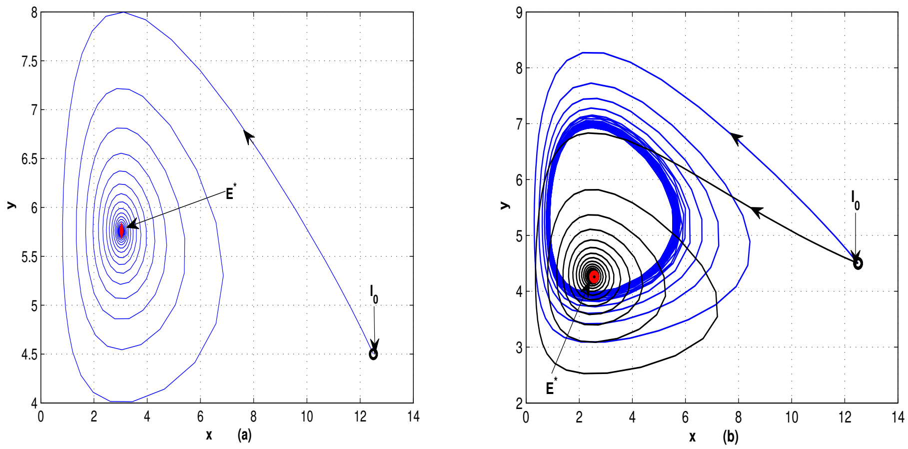

Figure 1: **(a)** The equilibrium point E^* is stable for the set of parametric values. **(b)** The figure depicts oscillatory behavior around at E^* of system (1) for $k_1 = 0.66$ and $K = 0.1$(blue line), stable behaviour at E^* for $k_1 = 0.66$ and $K = 0.2$(black line).

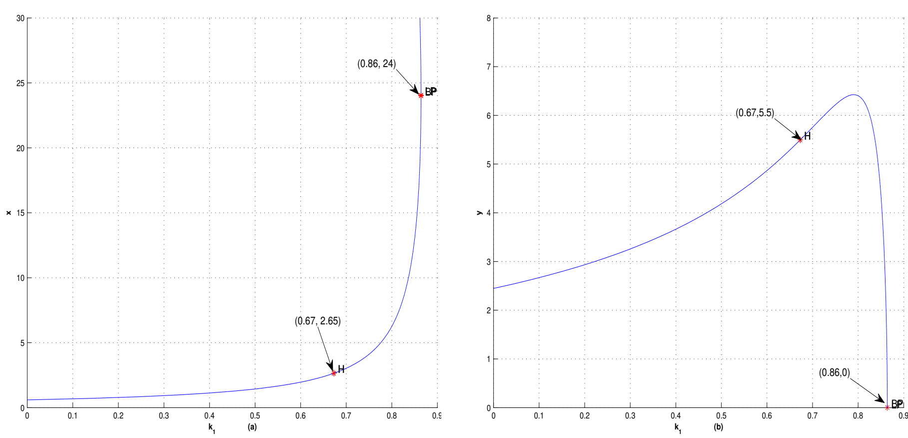

Figure 2: **(a-b)** The trajectory represents the different dynamical behaviors of prey and predator respectively for k_1.

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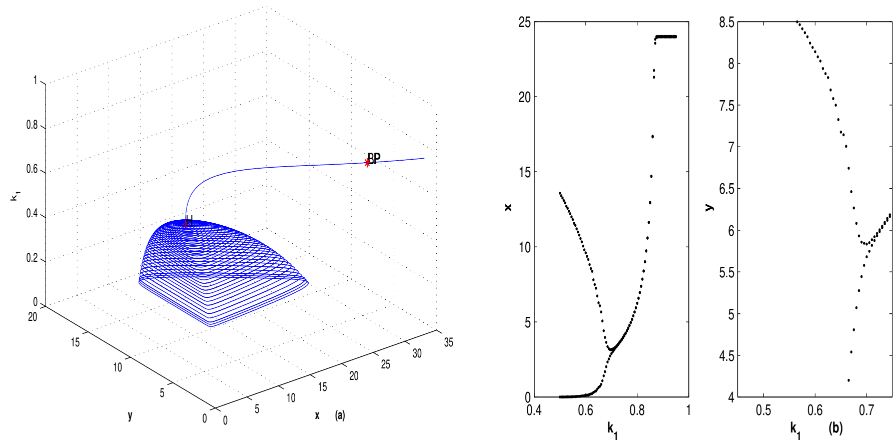

Figure 3: **(a)** The trajectory represents a family of stable limit cycles generate from Hopf (H) point for k_1 in $x-y-k_1$ plane. **(b)** Bifurcation diagram for k_1.

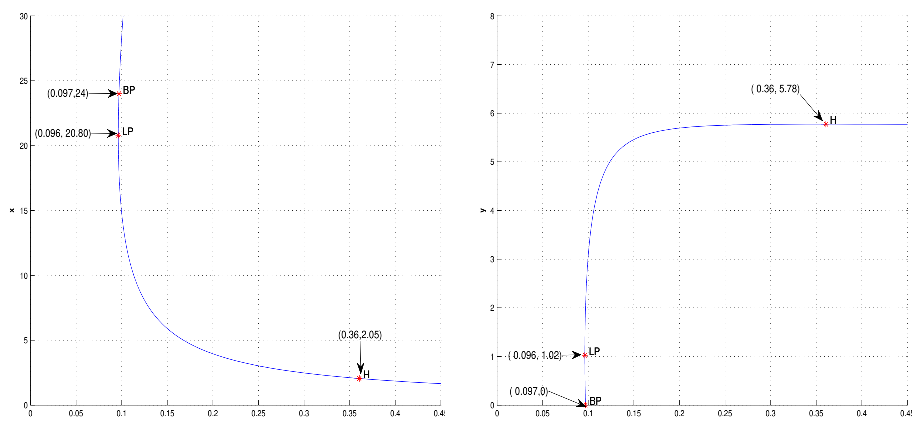

Figure 4: **(a-b)** The trajectory represents the different dynamical behaviors of prey and predator respectively for e.

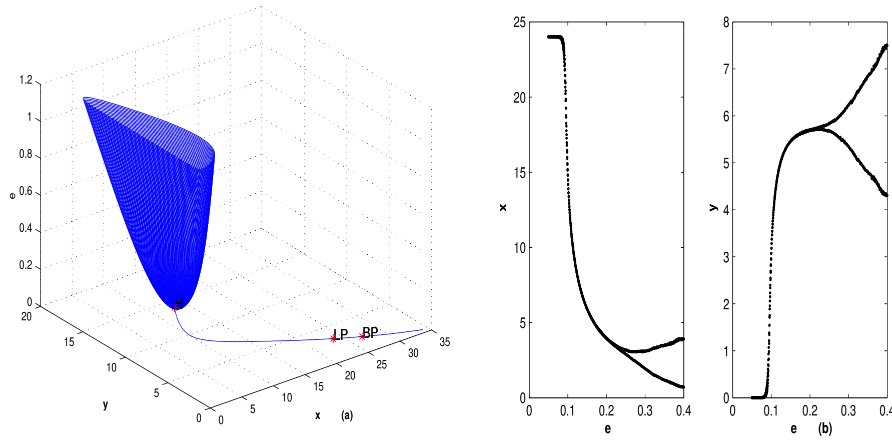

Figure 5: **(a)** The trajectory represents a family of stable limit cycles generate from Hopf (H) point for e in $x - y - e$ plane. **(b)** Bifurcation diagram for e.

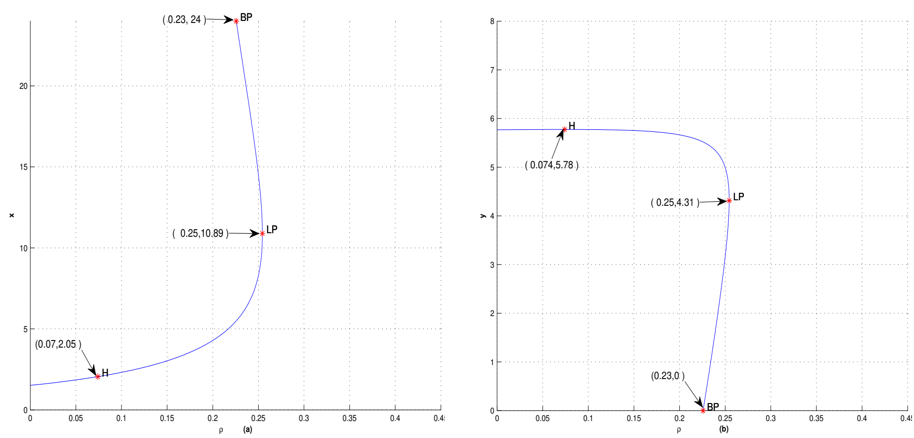

Figure 6: **(a-b)** The trajectory represents the different dynamical behaviors of prey and predator respectively for ρ.

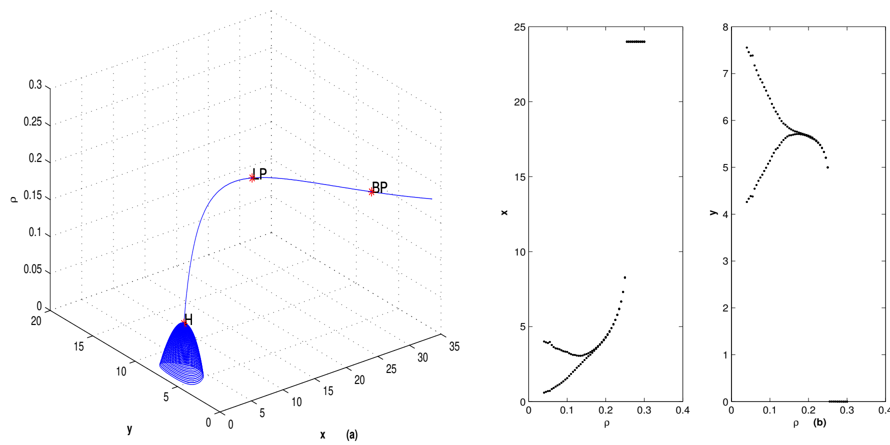

Figure 7: **(a)** The trajectory represents a family of stable limit cycles generate from Hopf (H) point for e in $x - y - \rho$ plane.. **(b)** Bifurcation diagram for ρ.

can increase both densities of prey and predator. On the other hand, when k_1 reaches a high risk threshold of the prey refuge the predator goes to extinct and the equilibrium E_1 is globally asymptotically stable.

6.2 Effect of e

Fig. 4(a-b) indicates that predator's conversion rate (e) play a crucial role to switch the prey and predator natures. Here, we have one Hopf point ($e = 0.360577$), Branch point ($e = 0.097047$) and a Limit point ($e = 0.096319$). Further, the system experiences a family of stable limit cycle generate from Hopf point (viz. Fig. 5(a)).

6.3 Effect of ρ

It is observed that the prey patches play a big impact in the system (1). From Fig. 6(a-b) & Fig. 7(a) it follow several stability behaviour and family of stable limit cycle for the free parameter ρ respectively. At $\rho = 1.416971$, the system experiences a super critical bifurcation with first Lyapunov coefficient $-2.031921e^{-03}$ and predator becomes extinct at $\rho = 0.225770$ i.e., at BP point. Also, a Limit point (LP) is obtained at $\rho = 0.254407$.

6.4 Bifurcation

The bifurcation diagrams (cf. Fig. 3(b), Fig. 5(b) and Fig. 7(b)) illustrate the complete dynamic pictures of the system (1) for the effect of parameter k_1, e

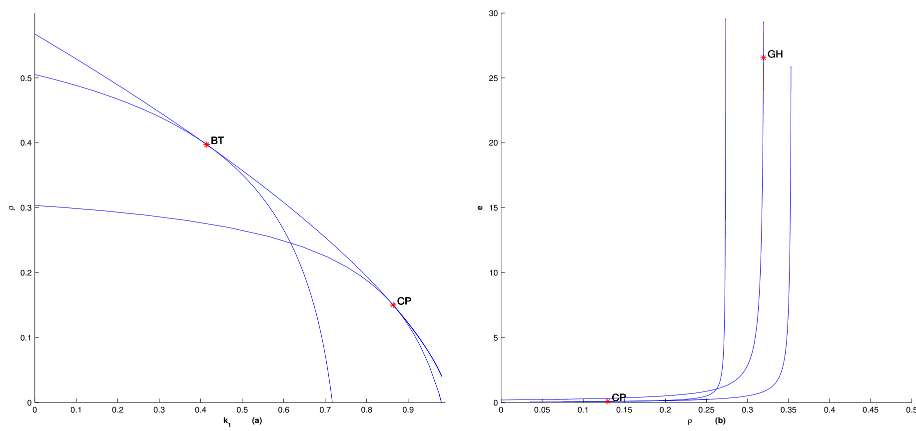

Figure 8: **(a)** Two parameters bifurcation diagram for $k_1 - \rho$. **(b)** Two parameters bifurcation diagram for $\rho - e$.

and ρ respectively. Fig. 5(a-b) display the two parameters bifurcation diagram for $k_1 - \rho$ and $\rho - e$ respectively. In this case, we see a Bogdanov-Takens (BT), Cusp bifurcation (CP) and Generalized Hopf (GH). **Generalized Hopf separates branches of sub-and supercritical Andronov-Hopf bifurcations in the two parameter plain**. The It is clearly indicates that a saddle-node bifurcation curve meet at transcritical curve at Cusp point(CP), i.e., **SN-TC** point and **saddle-node and Hopf bifurcation curve touch at** BT **point**. Also, the bifurcation curve exhibits a Generalized Hopf point (GH) where the 1^{st} Lyapunov coefficient turn out to be zero. All the numerical finding are summarized in Table 1.

7 Discussion

In this present article, a prey-predator model is designed by incorporating patches, prey refuge and fear effect to discover the dynamics of prey-predator systems. **It is assumed that prey population grows logistically and predators consume prey population under Holling II functional response.** Firstly, some basic properties are analyzed and verified which are ecologically well behaved such as boundedness and properties of existence of equilibria. **The local stability behavior of the system is carried out around each equilibrium. In order to explore the dynamics of proposed system, it is identified that, the system (1) has two equilibrium point such as axial (E_1) and coexistence equilibrium (E^*). We also perform the global stability of coexistence equilibrium by choosing a suitable Lyapunov function.** Throughout the analysis, availability of prey, i.e., the parameter k_1 play crucial role to exhibit Hopf bifurcation and stability

Table 1: Natures of equilibrium points.

Parameters	Values	Eigenvalues	Equilibrium points
k_1	0.673026	$(\pm 0.284862i)$	Hopf (H)
	0.864180	$(0, -1.2029)$	Limit Point (LP)
	0.864180	$(0, -1.2029)$	Branch Point (BP)
e	0.360577	$(\pm 0.305907i)$	Hopf (H)
	0.096319	$(0, -1.00886)$	Limit Point(LP)
	0.097047	$(0, -1.2)$	Branch Point(BP)
ρ	0.074021	$(\pm 0.289879i)$	Hopf (H)
	0.225770	$(0, -1.2)$	Branch Point(BP)
	0.254407	$(0, -0.398958)$	Limit Point(BP)
(k_1, ρ)	$(0.4146440.397248)$	$(\approx \pm 0.00)$	Bogdanov-Takens (BT)
	$(0.8639100.150199)$	$(0, -1.20027)$	Cusp bifurcation (CP)
(ρ, e)	$(0.0835480.129990)$	$(0, -1.2)$	Cusp bifurcation (CP)
	$(0.319445, 26.549989)$	$(\pm 1.53468i)$	Generalized Hopf (GH)

switching behavior. Numerically, we observe that when $k_1 < k_1^c = 0.673026$, the system exhibits oscillatory behavior and each population shows stable coexistence between $0.673026 < k_1 < 0.864180$. When processed further, coexistence equilibrium looses stability via transcritical bifurcation i.e., branch point and the predator population will die out. Similar characteristic nature of prey and predator have been seen for the effect of conversion rate of predator and toxicity level due to patches. Further, to study the impact of fear effect, prey shows anti-predator behaviours. **Several two parameter bifurcations are drawn that show different stability nature of dynamics.** It is observed that high value of fear level can stabilize the whole system in presence of high availability of prey species for predation. **So, availability of prey species, conversion rate of predator, prey patches and fear level acts an crucial roles in in determining the long-term population dynamics. We hope that this study will contribute in understanding the impact of fear, effect of conversion rate of predator and toxicity level due to patches. The system (1) can also be modified further for two prey and one or two predator which may be more significant to the biological diversity.**

8 Conclusion

In this article, we consider fear effect prey-predator model and a prey refuge with forming patches. By examining the characteristic equation of the corresponding linearized system we obtain the threshold conditions for the stability of system. It is observed that level of fear, availability of prey due to refuge mechanism, conversion rate of predator and toxicity level due to patches play major role to stabilize the system. We find that combined effects of more than one parameters results in complex dynamical behaviour.

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Non-polynomial fractal quintic spline method for nonlinear boundary-value problems

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Abstract

In this study, we have proposed second, fourth and sixth order convergent numerical techniques for approximating linear and non-linear boundary value problems of second order with the help of fractal non-polynomial spline function. We have discussed the convergence analysis and error bound for sixth order method to prove the theoretical aspects of the presented method. Numerical problems are experimented to validate the theoretical results. Comparison with fractal polynomial and few other existing methods leads us to the conclusion that the proposed technique is more efficient.

Keywords: Difference equations, fractal non-polynomial spline, quasilinearisation, convergence analysis, truncation error.

Mathematics Subject Classification: 28A80, 65D07, 34B15

1. Introduction

With the help of fractal non-polynomial spline, we have developed numerical techniques to find the approximate solution of boundary value problems(BVPs) of the type:

$$\begin{cases} w_{tt}(t)+\mathtt{p}(t)w(t)=\mathtt{f}(t), & t\in(0,\ 1), \\ w(0)=\sigma_0, & w(1)=\sigma_1, \end{cases} \tag{1.1}$$

and

$$\begin{cases} w_{tt}(t)+\mathtt{F}(t,w(t))=0, & t\in(0,\ 1), \\ w(0)=\sigma_0, & w(1)=\sigma_1, \end{cases} \tag{1.2}$$

where σ_0 and σ_1 are constants. In (1.1), $\mathtt{p}(t)$ and $\mathtt{f}(t)$ are continuous functions in closed interval $I=[0,\ 1]$. For random choices of $\mathtt{p}$ and $\mathtt{f}$, exact solution of these BVPs cannot be find.

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Therefore we approach numerical methods to get approximate solution of (1.1). In (1.2), presume that for $(t, w(t)) \in \mathtt{D} = \{0 \le t \le 1,\ -\infty < w(t) < \infty\}$, $\mathtt{F}$ and $\frac{\partial \mathtt{F}}{\partial w}$ are continuous. We know that (1.2) admits unique solution, if $\sup\limits_{(t,w(t))\in D} \frac{\partial \mathtt{F}}{\partial w} < \pi^2$,[22]. Here we assume that $\frac{\partial \mathtt{F}}{\partial w} \le 0$ on D and $\frac{\partial \mathtt{F}}{\partial w} < 0$ on $D^* = \{0 < t < 1,\ -\infty < w(t) < \infty\}$. The notation w_{tt} symbolizes second derivative of w with respect to t.

Various authors have used different techniques to find numerical solution of linear as well as non-linear BVPs. Authors in [11] used cubic spline functions to find the approximate solution of nonlinear BVPs. Few numerical techniques derived by various authors for solving non-linear BVPs are given in [1, 2, 8, 14, 23, 27, 28, 32] and fractional differential equations are given in [13, 15, 16, 17, 18, 19, 29, 30].

With the help of quasilinearisation technique[6, 21, 26], the non-linear BVP (1.2) is converted into a system of linear BVPs, which in turn are solved by derived numerical scheme using fractal non-polynomial quintic spline function. A parameter λ called scaling factor is used in fractal spline which is suitably restricted to obtain the approximate solution of the linearized BVPs. Fractal interpolation function was introduced by Barnsley[4] using Iterated function system. Although fractals are difficult to constrain but they are best suitable for generation of various irregular shapes found in nature. It provides the possibility of simulating and describing landscapes precisely with the help of mathematical models. To find the numerical solution of (1.2), Balasubramani et. al.[3] have worked upon fractal quintic polynomial spline functions. In this paper we have worked upon finding the approximate solution using fractal non-polynomial spline functions and observed that the proposed scheme provides better results. The description of paper is as follows:

At the beginning ,we have given a brief description of the presented method which uses fractal non-polynomial quintic spline to get a relation between $w(t)$ and $M(t)$ using continuity conditions. In section 3, we have discussed the truncation error. Thereafter, possible classes of method are discussed in section 4. Then we have discussed the convergence analysis of sixth order method in section 5. Error bounds are carried out. Thereafter, we have given a briefing about finite-difference method and Numerov's method, and experimented four numerical problems to testify the efficacy of proposed method in section 6. Concluding remarks are provided in section 7.

2. Fractal Nonpolynomial spline

Let $0 = t_0 < t_1 < t_2 < \ldots < t_n = 1$ be the partition of the interval $I = [0, 1]$ given in (1.1) and (1.2). Let $w(t)$ and $W_{\mathtt{j}}$ denote the analytical and approximate solutions respectively. For $t_{\mathtt{j}} = \mathtt{j}h,\ h = 1/n,\ \mathtt{j} = 0, 1, \ldots, n$. Let $M_{\mathtt{j}}$ and $S_{\mathtt{j}}$ denote the approximation corresponding to $w_{tt}(t_{\mathtt{j}})$ and $w_{tttt}(t_{\mathtt{j}})$ respectively.

Concept of Iterated functions system (IFS) is used to develop fractal interpolation functions(FIF). Basic details related to fractal interpolation are provided in [5, 9, 10].

Define $\mathtt{H_j} : I \to I_{\mathtt{j}}$, where $I_{\mathtt{j}} = [t_{\mathtt{j}-1}, t_{\mathtt{j}}]$ such that

$$\mathtt{H_j}(t) = ht + t_{\mathtt{j}-1},\ t \in I.$$

Clearly, $\mathtt{H_j}(t_0) = t_{\mathtt{j}-1}$ and $\mathtt{H_j}(t_n) = t_{\mathtt{j}}$,
and define $\mathbb{F}_{\mathtt{j}} : I \times \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{F}_{\mathtt{j}}(t,w) = \lambda w + r_{\mathtt{j}}(t),\ (t,w)\ \in\ I \times \mathbb{R},$$

where λ is scaling factor such that $|\lambda| < h^4$ and

$$r_{\mathtt{j}}(t) = \mathtt{A_j} cos\xi(t-t_0) + \mathtt{B_j} sin\xi(t-t_0) + \mathtt{C_j}(t-t_0)^3 + \mathtt{D_j}(t-t_0)^2 + \mathtt{E_j}(t-t_0) + \mathtt{F_j}.$$

Constructing the IFS as follows

$$I \times \mathbb{R}; X_{\mathtt{j}}(t,w) = (\mathtt{H_j}(t), (\mathbb{F}_{\mathtt{j}}(t,w))) : \mathtt{j} = 1,2,\ldots,n,$$

which satisfies the following conditions:

$$\begin{cases} \mathbb{F}_{\mathtt{j}}(t_0, W_0) = W_{\mathtt{j}-1},\ \ \mathbb{F}_{\mathtt{j}}(t_n, W_n) = W_{\mathtt{j}}, \\ \mathbb{F}_{\mathtt{j},1}(t_n, W_{n,1}) = \mathbb{F}_{\mathtt{j}+1,1}(t_0, W_{0,1}), \\ \mathbb{F}_{\mathtt{j},2}(t_0, M_0) = M_{\mathtt{j}-1},\ \ \mathbb{F}_{\mathtt{j},2}(t_n, M_n) = M_{\mathtt{j}}, \\ \mathbb{F}_{\mathtt{j},3}(t_n, W_{n,3}) = \mathbb{F}_{\mathtt{j}+1,3}(t_0, W_{0,3}), \\ \mathbb{F}_{\mathtt{j},4}(t_0, S_0) = S_{\mathtt{j}-1},\ \ \mathbb{F}_{\mathtt{j},4}(t_n, S_n) = S_{\mathtt{j}}, \end{cases}$$

where $\mathtt{j} = 1,2,....,n-1$, and $\mathbb{F}_{\mathtt{j},\mathtt{k}}(t,w) = \frac{\lambda w + r_{\mathtt{j}}^{\mathtt{k}}(t)}{h^{\mathtt{k}}}$, $\mathtt{k} = 1,2,3,4$ and

$$W_{0,1} = \frac{r_1^{(1)}(t_0)}{h-\lambda},\ \ W_{n,1} = \frac{r_n^{(1)}(t_n)}{h-\lambda},\ W_{0,3} = \frac{r_1^{(3)}(t_0)}{h^3-\lambda},\ \ W_{n,3} = \frac{r_n^{(3)}(t_n)}{h^3-\lambda}.$$

Clearly, IFS is satisfying C^4-differentiability conditions on FIFs[5, 9, 10].

Let $\mathcal{F} = \{\Phi\ \in\ \mathtt{C}^4(I,\mathbb{R}) \mid \Phi(t_0) = W_0,\ \Phi(t_n) = W_n,\ \Phi^{(2)}(t_0) = M_0,$
$\Phi^{(2)}(t_n) = M_n,\ \Phi^{(4)}(t_0) = S_0,\ \Phi^{(4)}(t_n) = S_n\}.$
Then $(\mathcal{F}, d)$ is a complete metric space and d is a metric induced on $\mathcal{F}$ by $\mathtt{C}^4-$norm. Let us define the Read-Bajraktarevic operator $\mathbb{T}$ on $(\mathcal{F}, d)$ as

$$\mathbb{T}(\Phi(\mathtt{H_j}(t))) = \lambda\Phi(t) + \mathtt{A_j} cos\xi(t-t_0) + \mathtt{B_j} sin\xi(t-t_0) + \mathtt{C_j}(t-t_0)^3 + \mathtt{D_j}(t-t_0)^2 + \mathtt{E_j}(t-t_0) + \mathtt{F_j}, \qquad t \in [t_0, t_n], \qquad \mathtt{j} = 1,2,....,n.$$

As operator $\mathbb{T}$ is contraction map, it must have a unique fixed point φ (say) which will satisfy the following conditions:

$$\varphi(\mathtt{H_j}(t)) = \lambda\varphi(t) + \mathtt{A_j} cos\xi(t-t_0) + \mathtt{B_j} sin\xi(t-t_0) + \mathtt{C_j}(t-t_0)^3 + \mathtt{D_j}(t-t_0)^2$$

$$+\mathtt{E_j}(t-t_0)+\mathtt{F_j}, \quad t\in[t_0,t_n], \;\; \mathtt{j}=1,2,....,n. \tag{2.1}$$

From [10], it can be seen that

$$\begin{cases} \mathbb{F}_{\mathtt{j}}(t_0,W_0)=W_{\mathtt{j}-1}, \;\; \mathbb{F}_{\mathtt{j}}(t_n,W_n)=W_{\mathtt{j}}, \;\; \mathbb{F}_{\mathtt{j},2}(t_0,M_0)=M_{\mathtt{j}-1}, \\ \mathbb{F}_{\mathtt{j},2}(t_n,M_n)=M_{\mathtt{j}}, \;\; \mathbb{F}_{\mathtt{j},4}(t_0,S_0)=S_{\mathtt{j}-1}, \;\; \mathbb{F}_{\mathtt{j},2}(t_n,S_n)=S_{\mathtt{j}}, \end{cases}$$

are equivalent to

$$\begin{cases} \varphi(t_{\mathtt{j}-1})=W_{\mathtt{j}-1}, \; \varphi(t_{\mathtt{j}})=W_{\mathtt{j}}, \; \varphi^{(2)}(t_{\mathtt{j}-1})=M_{\mathtt{j}-1}, \\ \varphi^{(2)}(t_{\mathtt{j}})=M_{\mathtt{j}}, \; \varphi^{(4)}(t_{\mathtt{j}-1})=S_{\mathtt{j}-1}, \; \varphi^{(4)}(t_{\mathtt{j}})=S_{\mathtt{j}}. \end{cases} \tag{2.2}$$

The conditions $\mathbb{F}_{\mathtt{j},1}(t_n,W_{n,1})=\mathbb{F}_{\mathtt{j}+1,1}(t_0,W_{0,1})$, and $\mathbb{F}_{\mathtt{j},3}(t_n,W_{n,3})=\mathbb{F}_{\mathtt{j}+1,3}(t_0,W_{0,3})$, can be reevaluated as $\varphi^{(1)}(\mathtt{H_j}(t_n))=\varphi^{(1)}(\mathtt{H_{j+1}}(t_0))$ and $\varphi^{(3)}(\mathtt{H_j}(t_n))=\varphi^{(3)}(\mathtt{H_{j+1}}(t_0))$ respectively. The coefficients $\mathtt{A}_j$, $\mathtt{B}_j$, $\mathtt{C}_j$, $\mathtt{D}_j$, $\mathtt{E}_j$ and $\mathtt{F}_j$ used in (2.1) are evaluated using (2.2). We get

$$\begin{aligned}
&\mathtt{A_j}=\tfrac{h^4}{\xi^4}\Big(S_{\mathtt{j}-1}-\tfrac{\lambda}{h^4}S_0\Big),\\
&\mathtt{B_j}=\tfrac{h^4}{\xi^4\sin\xi}\Big(S_{\mathtt{j}}-\tfrac{\lambda}{h^4}S_n\Big)-\tfrac{h^4\cos\xi}{\xi^4\sin\xi}\Big(S_{\mathtt{j}-1}-\tfrac{\lambda}{h^4}S_0\Big),\\
&\mathtt{C_j}=\tfrac{h^2}{6}\Big(M_{\mathtt{j}}-\tfrac{\lambda}{h^2}M_n\Big)-\tfrac{h^2}{6}\Big(M_{\mathtt{j}-1}-\tfrac{\lambda}{h^2}M_0\Big)+\tfrac{h^4}{6\xi^2}\Big(S_{\mathtt{j}}-\tfrac{\lambda}{h^4}S_n\Big)-\tfrac{h^4}{6\xi^2}\Big(S_{\mathtt{j}-1}-\tfrac{\lambda}{h^4}S_0\Big),\\
&\mathtt{D_j}=\tfrac{h^2}{2}\Big(M_{\mathtt{j}-1}-\tfrac{\lambda}{h^2}M_0\Big)+\tfrac{h^4}{2\xi^2}\Big(S_{\mathtt{j}-1}-\tfrac{\lambda}{h^4}S_0\Big),\\
&\mathtt{E_j}=\Big(W_{\mathtt{j}}-\lambda W_n\Big)-\Big(W_{\mathtt{j}-1}-\lambda W_0\Big)-\tfrac{h^4}{6\xi^4}(6+\xi^2)\Big(S_{\mathtt{j}}-\tfrac{\lambda}{h^4}S_n\Big)+\tfrac{h^4}{6\xi^4}(6-2\xi^2)\Big(S_{\mathtt{j}-1}-\tfrac{\lambda}{h^4}S_0\Big)\\
&\qquad-\tfrac{h^2}{6}\Big(M_{\mathtt{j}}-\tfrac{\lambda}{h^2}M_n\Big)-\tfrac{2h^2}{6}\Big(M_{\mathtt{j}-1}-\tfrac{\lambda}{h^2}M_0\Big),\\
&\mathtt{F_j}=\Big(W_{\mathtt{j}-1}-\lambda W_0\Big)+\tfrac{h^4}{\xi^4}\Big(S_{\mathtt{j}-1}-\tfrac{\lambda}{h^4}S_0\Big).
\end{aligned}$$

For continuity of $\varphi^{(1)}$, we have used $\varphi^{(1)}(t_{\mathtt{j}}^{-})=\varphi^{(1)}(t_{\mathtt{j}}^{+})$ i.e., $\varphi^{(1)}(\mathtt{H_j}(t_n))=\varphi^{(1)}(\mathtt{H_{j+1}}(t_0))$ and eventually get the following condition:

$$\lambda\varphi^{(1)}(t_n)-\mathtt{A_j}\xi\sin\xi+\mathtt{B_j}\xi\cos\xi+3\mathtt{C_j}+2\mathtt{D_j}+\mathtt{E_j}=\lambda\varphi^{(1)}(t_0)+\xi\mathtt{B_{j+1}}+\mathtt{E_{j+1}}. \tag{2.3}$$

Similarly for continuity of $\varphi^{(3)}$ we have used $\varphi^{(3)}(t_{\mathtt{j}}^{-})=\varphi^{(3)}(t_{\mathtt{j}}^{+})$ i.e., $\varphi^{(3)}(\mathtt{H_j}(t_n))=\varphi^{(3)}(\mathtt{H_{j+1}}(t_0))$ and get

$$\lambda\varphi^{(3)}(t_n)+\mathtt{A_j}\xi^3\sin\xi-\mathtt{B_j}\xi^3\cos\xi+6\mathtt{C_j}=\lambda\varphi^{(3)}(t_0)+\xi^3\mathtt{B_{j+1}}+6\mathtt{C_{j+1}}. \tag{2.4}$$

After substituting the values of $\mathtt{A_j}$, $\mathtt{B_j}$, $\mathtt{C_j}$, $\mathtt{D_j}$, $\mathtt{E_j}$, $\mathtt{B_{j+1}}$, $\mathtt{C_{j+1}}$ and $\mathtt{E_{j+1}}$ in (2.3) and (2.4), we obtain

$$\Big(S_0+S_n\Big)\Big(\frac{\lambda}{2\xi^2}+\frac{\lambda}{\xi^3}\frac{\cos\xi}{\sin\xi}-\frac{\lambda}{\xi^3}\sin\xi\Big)+\Big(S_{\mathtt{j}-1}+S_{\mathtt{j}+1}\Big)\Big(\frac{h^4}{\xi^3\sin\xi}-\frac{h^4}{6\xi^4}(6+\xi^2)\Big)$$

$$+S_{\mathrm{j}}\Big(\frac{h^4}{6\xi^4}(12-4\xi^2)-\frac{2h^4}{\xi^3}\frac{\cos\xi}{\sin\xi}\Big)=\lambda\varphi^{(1)}(t_n)-\lambda\varphi^{(1)}(t_0)-(W_{\mathrm{j}+1}-2W_{\mathrm{j}}+W_{\mathrm{j}-1})$$
$$-\frac{\lambda}{2}(M_0+M_n)+\frac{h^2}{6}(M_{\mathrm{j}+1}+4M_{\mathrm{j}}+M_{\mathrm{j}-1}), \tag{2.5}$$

and

$$(S_0+S_n)\Big(\frac{\lambda}{\xi}\frac{\cos\xi}{\sin\xi}-\frac{\lambda}{\xi\sin\xi}\Big)+\Big(\frac{h^4}{\xi\sin\xi}-\frac{h^4}{\xi^2}\Big)(S_{\mathrm{j}-1}+S_{\mathrm{j}+1})+S_{\mathrm{j}}\Big(\frac{2h^4}{\xi^2}-\frac{2h^4}{\xi}\frac{\cos\xi}{\sin\xi}\Big)$$
$$=\lambda(\varphi^{(3)}(t_0)-\varphi^{(3)}(t_n))+h^2(M_{\mathrm{j}-1}-2M_{\mathrm{j}}+M_{\mathrm{j}+1}), \tag{2.6}$$

respectively. From equation (2.5), we have

$$\Big(\alpha_2S_{\mathrm{j}-1}+2\beta_2S_{\mathrm{j}}+\alpha_2S_{\mathrm{j}+1}\Big)=-\frac{1}{6h^2}(M_{\mathrm{j}+1}+4M_{\mathrm{j}}+M_{\mathrm{j}-1})+\frac{1}{h^4}k_2\Big(S_0+S_n\Big)-\frac{\lambda}{h^4}\big(\varphi^{(1)}(t_n)$$
$$-\varphi^{(1)}(t_0)\big)+\frac{\lambda}{2h^4}(M_0+M_n)+\frac{1}{h^4}(W_{\mathrm{j}+1}-2W_{\mathrm{j}}+W_{\mathrm{j}+1}), \tag{2.7}$$

and from equation (2.6), we have

$$(\alpha_1S_{\mathrm{j}-1}+2\beta_1S_{\mathrm{j}}+\alpha_1S_{\mathrm{j}+1})=\frac{1}{h^2}(M_{\mathrm{j}+1}-2M_{\mathrm{j}}+M_{\mathrm{j}-1})-\frac{1}{h^4}k_1(S_0+S_n)$$
$$-\frac{\lambda}{h^4}\big(\varphi^{(3)}(t_n)-\varphi^{(3)}(t_0)\big), \tag{2.8}$$

where
$\alpha_1=\frac{1}{\xi^2}\Big(\xi cosec(\xi)-1\Big)$, $\beta_1=\frac{1}{\xi^2}\Big(1-\xi cot(\xi)\Big)$,
$\alpha_2=\frac{1}{\xi^2}\Big(\frac{1}{6}-\alpha_1\Big)$, $\beta_2=\frac{1}{\xi^2}\Big(\frac{1}{3}-\beta_1\Big)$,
$k_1=\frac{cot\xi}{\xi}-\frac{cosec\xi}{\xi}$, $k_2=\frac{1}{\xi^2}\Big(\frac{1}{2}+k_1\Big)$.
Solving (2.7) and (2.8), we get

$$S_{\mathrm{j}}=\frac{(S_0+S_n)}{2h^4}\frac{(\alpha_1k_2+\alpha_2k_1)}{(\alpha_1\beta_2-\alpha_2\beta_1)}-\frac{\alpha_1\lambda}{2h^4}\frac{\big(\varphi^{(1)}(t_n)-\varphi^{(1)}(t_0)\big)}{(\alpha_1\beta_2-\alpha_2\beta_1)}+\frac{\alpha_2\lambda}{2h^4}\frac{\big(\varphi^{(3)}(t_n)-\varphi^{(3)}(t_0)\big)}{(\alpha_1\beta_2-\alpha_2\beta_1)}$$
$$+\frac{\alpha_1\lambda}{4h^4}\frac{(M_0+M_n)}{(\alpha_1\beta_2-\alpha_2\beta_1)}+\frac{\alpha_1}{2h^4}\frac{(W_{\mathrm{j}+1}-2W_{\mathrm{j}}+W_{\mathrm{j}-1})}{(\alpha_1\beta_2-\alpha_2\beta_1)}-\frac{\alpha_1}{12h^2}\frac{(M_{\mathrm{j}+1}+4M_{\mathrm{j}}+M_{\mathrm{j}-1})}{(\alpha_1\beta_2-\alpha_2\beta_1)}$$
$$-\frac{\alpha_2}{2h^2}\frac{(M_{\mathrm{j}+1}-2M_{\mathrm{j}}+M_{\mathrm{j}-1})}{(\alpha_1\beta_2-\alpha_2\beta_1)}. \tag{2.9}$$

Using equation (2.9) in equation (2.8), we have

$$\alpha_1(W_{\mathrm{j}+2}+W_{\mathrm{j}-2})+2(\beta_1-\alpha_1)(W_{\mathrm{j}+1}+W_{\mathrm{j}-1})+(2\alpha_1-4\beta_1)W_{\mathrm{j}}$$
$$=-2(\alpha_1+\beta_1)\lambda(\varphi^{(1)}(t_0)-\varphi^{(1)}(t_n))+2(\alpha_2+\beta_2)\lambda(\varphi^{(3)}(t_0)-\varphi^{(3)}(t_n))$$
$$-(\alpha_1+\beta_1)\lambda(M_0+M_n)+h^2(pM_{\mathrm{j}+2}+qM_{\mathrm{j}+1}+rM_{\mathrm{j}}+qM_{\mathrm{j}-1}+pM_{\mathrm{j}-2}), \tag{2.10}$$

where
$p=\alpha_2+\frac{\alpha_1}{6}$,
$q=2\Big[\frac{1}{6}(2\alpha_1+\beta_1)-(\alpha_2-\beta_2)\Big]$,
$r=2\Big[\frac{1}{6}(\alpha_1+4\beta_1)+(\alpha_2-2\beta_2)\Big]$.

Remark 1: When $\Big(\alpha_1,\beta_1,\alpha_2,\beta_2\Big)=\Big(\frac{1}{6},\frac{2}{6},\frac{-7}{360},\frac{-8}{360}\Big)$ equation (2.10) reduces to (2.5) of Balasubramani et al.[3].

Remark 2: When $\lambda=0$, equation (2.10) reduces to quintic non-polynomial spline method by P. Srivastav et al.[31].

2.1. Spline Solution for Linear BVPs

Equation (1.1) is discretized at $t=t_{\mathtt{j}}$, since $M_{\mathtt{j}}+\mathtt{p}_{\mathtt{j}}W_{\mathtt{j}}=\mathtt{f}_{\mathtt{j}}$, where $\mathtt{p}_{\mathtt{j}}=\mathtt{p}(t_{\mathtt{j}})$, $\mathtt{f}_{\mathtt{j}}=\mathtt{f}(t_{\mathtt{j}})$. The boundary equations are discretized as $W_0=\sigma_0,\ \ W_n=\sigma_1$.
Substitute

$$\varphi^{(3)}(t_0)=\frac{-W_0+3W_1-3W_2+W_3}{h^3},\qquad \varphi^{(3)}(t_n)=\frac{W_n-3W_{n-1}+3W_{n-2}-W_{n-3}}{h^3},$$
$$\varphi^{(1)}(t_0)=\frac{W_1-W_0}{h},\qquad \varphi^{(1)}(t_n)=\frac{W_n-W_{n-1}}{h},$$
$$M_{\mathtt{j}}=\mathtt{f}_{\mathtt{j}}-\mathtt{p}_{\mathtt{j}}W_{\mathtt{j}},$$

in (2.10), and after some calculations we get,

$$\begin{cases}
-\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_1+\Big[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_2-\Big[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_3-\Big[\alpha_1+ph^2\mathtt{p}_{\mathtt{j}-2}\Big]W_{\mathtt{j}-2}\\
-\Big[2(\beta_1-\alpha_1)+qh^2\mathtt{p}_{\mathtt{j}-1}\Big]W_{\mathtt{j}-1}-\Big[(2\alpha_1-4\beta_1)+rh^2\mathtt{p}_{\mathtt{j}}\Big]W_{\mathtt{j}}-\Big[2(\beta_1-\alpha_1)\\
+qh^2\mathtt{p}_{\mathtt{j}+1}\Big]W_{\mathtt{j}+1}-\Big[\alpha_1+ph^2\mathtt{p}_{\mathtt{j}+2}\Big]W_{\mathtt{j}+2}-\Big[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_{n-3}+\Big[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_{n-2}\\
-\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_{n-1}=-h^2\Big[p(\mathtt{f}_{\mathtt{j}+2}+\mathtt{f}_{\mathtt{j}-2})+q(\mathtt{f}_{\mathtt{j}+1}+\mathtt{f}_{\mathtt{j}-1})+r\mathtt{f}_{\mathtt{j}}\Big]\\
+\lambda(\alpha_1+\beta_1)[(\mathtt{f}_0+\mathtt{f}_n)-(\mathtt{p}_0\sigma_0+\mathtt{p}_n\sigma_n)]-\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]\sigma_0\\
-\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]\sigma_1,\qquad\qquad \mathtt{j}=2,3,\ldots,(n-2).
\end{cases}\tag{2.11}$$

In (2.11) we have $(n-1)$ unknowns $W_1,\ W_2,\ldots W_{n-1}$ and $(n-3)$ equations. Therefore two more equations are required to find unique solution. Hence we derive two boundary equations as follows:

Boundary equations

Let the equation at $\mathtt{j}=1$ and $\mathtt{j}=n-1$ be

$$\begin{cases}
\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_0-\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_1+\Big[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_2\\
-\Big[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_3-\Big[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_{n-3}+\Big[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_{n-2}-\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h}\\
+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_{n-1}+\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_n=\lambda(\alpha_1+\beta_1)[(\mathtt{f}_0+\mathtt{f}_n)\\
-(q_0\sigma_0+q_n\sigma_n)]+\sum_{\mathtt{k}=0}^{\mathtt{k}=5}\big(l_{\mathtt{k}}w(t_{\mathtt{k}})+\mathtt{m}_{\mathtt{k}}h^2w_{tt}(t_{\mathtt{k}})\big),
\end{cases}\tag{2.12}$$

and

$$\begin{cases} \left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_0-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_1+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_2 \\ -\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_3-\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-3}+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-2}-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}\right. \\ \left.+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-1}+\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_n=\lambda(\alpha_1+\beta_1)[(\mathtt{f}_0+\mathtt{f}_n) \\ -(q_0\sigma_0+q_n\sigma_n)]+\sum_{\mathtt{k}=n-5}^{\mathtt{k}=n}\left(l_{\mathtt{k}}w(t_{\mathtt{k}})+\mathtt{m}_{\mathtt{k}}h^2w_{tt}(t_{\mathtt{k}})\right), \end{cases} \tag{2.13}$$

respectively. The system (2.11), (2.12) and (2.13) provides the numerical solution $W_{\mathtt{j}},\ \mathtt{j}=1,2,\ldots,n-1$ for linear BVPs.

2.2. Spline Solution for nonlinear BVPs

2.2.1. Quasilinearisation technique

We use quasilinearisation technique to convert the non-linear BVP given in (1.2) into a system of linear BVPs. Here $w^{(0)}(t)$ denotes the initial approximation and the function $\mathtt{F}(t,w(t))$ is expanded around the $w^{(0)}(t)$ to obtain

$$\mathtt{F}(t,w^{(1)}(t))=\mathtt{F}(t,w^{(0)}(t))+(w^{(1)}-w^{(0)})\big(\frac{\partial \mathtt{F}}{\partial w}\big)_{(t,w^{(0)}(t))}+\cdots.$$

In general,

$$\mathtt{F}(t,w^{(\mathtt{r}+1)}(t))=\mathtt{F}(t,w^{(\mathtt{r})}(t))+(w^{(\mathtt{r}+1)}-w^{(\mathtt{r})})\big(\frac{\partial \mathtt{F}}{\partial w}\big)_{(t,w^{(\mathtt{r})}(t))}+\cdots,$$

where $\mathtt{r}$ is the iteration index such that $\mathtt{r}=0,1,2,...$
The nonlinear BVP (1.2) can be written as

$$\begin{cases} w_{tt}^{(\mathtt{r}+1)}(t)+\mathtt{F}(t,w^{(\mathtt{r}+1)}(t))=0, \quad t\in(0,1), \\ w^{(\mathtt{r}+1)}(0)=\sigma_0, \quad w^{(\mathtt{r}+1)}(1)=\sigma_1. \end{cases} \tag{2.14}$$

By substituting

$$\mathtt{F}(t,w^{(\mathtt{r}+1)}(t))=\mathtt{F}(t,w^{(\mathtt{r})}(t))+(w^{(\mathtt{r}+1)}-w^{(\mathtt{r})})\big(\frac{\partial \mathtt{F}}{\partial w}\big)_{(t,w^{(\mathtt{r})}(t))}$$

in (2.14), we get

$$\begin{cases} w_{tt}^{(\mathtt{r}+1)}(t)+q^{(\mathtt{r})}(t)w^{(\mathtt{r}+1)}(t)=f^{(\mathtt{r})}(t), \quad t\in(0,1), \quad \mathtt{r}=0,1,..., \\ w^{(\mathtt{r}+1)}(0)=\sigma_0, \quad w^{(\mathtt{r}+1)}(1)=\sigma_1, \end{cases} \tag{2.15}$$

where

$$q^{(\mathtt{r})}(t)=\big(\frac{\partial \mathtt{F}}{\partial w}\big)_{(t,w^{(\mathtt{r})}(t))}, \quad f^{(\mathtt{r})}(t)=w^{(\mathtt{r})}(t)\big(\frac{\partial \mathtt{F}}{\partial w}\big)_{(t,w^{(\mathtt{r})}(t))}-\mathtt{F}(t,w^{(\mathtt{r})}(t)).$$

Hence the non-linear BVP (1.2) is converted into a system of linear BVPs. Now we will proceed to solve this system numerically.

2.2.2. **Numerical scheme**

Let $W_{\mathtt{j}}^{(\mathtt{r})}$ is the approximate value of $w^{(\mathtt{r})}(t_{\mathtt{j}})$ and $M_{\mathtt{j}}^{(\mathtt{r})}$ is the approximate value of $w_{tt}^{(\mathtt{r})}(t_{\mathtt{j}})$. Now, at $t = t_{\mathtt{j}}$, the differential equation (2.15) can be discretized as

$$M_{\mathtt{j}}^{(\mathtt{r}+1)} + q_{\mathtt{j}}^{(\mathtt{r})} W_{\mathtt{j}}^{(\mathtt{r}+1)} = f_{\mathtt{j}}^{(\mathtt{r})},$$

where

$$q_{\mathtt{j}}^{(\mathtt{r})} = \Big(\frac{\partial F}{\partial w}\Big)_{(t_{\mathtt{j}}, w_{\mathtt{j}}^{(\mathtt{r})})}, \quad f_{\mathtt{j}}^{(\mathtt{r})} = w_{\mathtt{j}}^{(\mathtt{r})} \Big(\frac{\partial F}{\partial w}\Big)_{(t_{\mathtt{j}}, w_{\mathtt{j}}^{(\mathtt{r})})} - \mathtt{F}(t_{\mathtt{j}}, w_{\mathtt{j}}^{(\mathtt{r})}).$$

Also, the boundary conditions can be discretised as $W_0^{(\mathtt{r}+1)} = \sigma_0, \quad W_n^{(\mathtt{r}+1)} = \sigma_1$.

Substitute

$$\varphi^{(3)}(t_0) = \frac{-W_0^{(\mathtt{r}+1)} + 3W_1^{(\mathtt{r}+1)} - 3W_2^{(\mathtt{r}+1)} + W_3^{(\mathtt{r}+1)}}{h^3},$$

$$\varphi^{(3)}(t_n) = \frac{W_n^{(\mathtt{r}+1)} - 3W_{n-1}^{(\mathtt{r}+1)} + 3W_{n-2}^{(\mathtt{r}+1)} - W_{n-3}^{(\mathtt{r}+1)}}{h^3},$$

$$\varphi^{(1)}(t_0) = \frac{W_1^{(\mathtt{r}+1)} - W_0^{(\mathtt{r}+1)}}{h}, \qquad \varphi^{(1)}(t_n) = \frac{W_n^{(\mathtt{r}+1)} - W_{n-1}^{(\mathtt{r}+1)}}{h},$$

$$M_{\mathtt{j}}^{(\mathtt{r}+1)} = f_{\mathtt{j}}^{(\mathtt{r})} - q_{\mathtt{j}}^{(\mathtt{r})} W_{\mathtt{j}}^{(\mathtt{r}+1)},$$

in equation (2.10) we have

$$\begin{cases} -\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h} + \frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big] W_1^{(\mathtt{r}+1)} + \Big[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big] W_2^{(\mathtt{r}+1)} - \Big[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big] W_3^{(\mathtt{r}+1)} - \Big[\alpha_1 \\ +ph^2 q_{\mathtt{j}-2}^{(\mathtt{r})}\Big] W_{\mathtt{j}-2}^{(\mathtt{r}+1)} - \Big[2(\beta_1-\alpha_1) + qh^2 q_{\mathtt{j}-1}^{(\mathtt{r})}\Big] W_{\mathtt{j}-1}^{(\mathtt{r}+1)} - \Big[(2\alpha_1 - 4\beta_1) + rh^2 q_{\mathtt{j}}^{(\mathtt{r})}\Big] W_{\mathtt{j}}^{(\mathtt{r}+1)} \\ -\Big[2(\beta_1-\alpha_1) + qh^2 q_{\mathtt{j}+1}^{(\mathtt{r})}\Big] W_{\mathtt{j}+1}^{(\mathtt{r}+1)} - \Big[\alpha_1 + ph^2 q_{\mathtt{j}+2}^{(\mathtt{r})}\Big] W_{\mathtt{j}+2}^{(\mathtt{r}+1)} - \Big[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big] W_{n-3}^{(\mathtt{r}+1)} \\ +\Big[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big] W_{n-2}^{(\mathtt{r}+1)} - \Big[\frac{2(\alpha_1+\beta_1)\lambda}{h} + \frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big] W_{n-1}^{(\mathtt{r}+1)} = -h^2\Big[p(f_{\mathtt{j}+2}^{(\mathtt{r})} + f_{\mathtt{j}-2}^{(\mathtt{r})}) \\ +q(f_{\mathtt{j}+1}^{(\mathtt{r})} + f_{\mathtt{j}-1}^{(\mathtt{r})}) + rf_{\mathtt{j}}^{(\mathtt{r})}\Big] + \lambda(\alpha_1+\beta_1)[(f_0^{(\mathtt{r})} + f_n^{(\mathtt{r})}) - (q_0^{(\mathtt{r})}\sigma_0 + q_n^{(\mathtt{r})}\sigma_n)] \\ -\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h} + \frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]\sigma_0 - \Big[\frac{2(\alpha_1+\beta_1)\lambda}{h} + \frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]\sigma_1, \quad \mathtt{j} = 2,3,\ldots,(n-2). \end{cases} \tag{2.16}$$

In (2.16) we have $(n-1)$ unknowns $W_1^{(\mathtt{r}+1)}, W_2^{(\mathtt{r}+1)}, \ldots W_{n-1}^{(\mathtt{r}+1)}$ and $(n-3)$ equations. Therefore two more equations are required to find unique solution. Hence we derive two boundary equations as follows:

Boundary equations

Let the equation at $\mathrm{j}=1$ and $\mathrm{j}=n-1$ be

$$\begin{cases} \left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_0^{(\mathrm{r}+1)}-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_1^{(\mathrm{r}+1)}+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_2^{(\mathrm{r}+1)} \\ -\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_3^{(\mathrm{r}+1)}-\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-3}^{(\mathrm{r}+1)}+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-2}^{(\mathrm{r}+1)}-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}\right. \\ \left.+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-1}^{(\mathrm{r}+1)}+\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_n^{(\mathrm{r}+1)}=\lambda(\alpha_1+\beta_1)[(f_0^{(\mathrm{r})}+f_n^{(\mathrm{r})}) \\ -(q_0^{(\mathrm{r})}\sigma_0+q_n^{(\mathrm{r})}\sigma_n)]+\sum_{\mathrm{k}=0}^{\mathrm{k}=5}\left(l_\mathrm{k}w^{(\mathrm{r}+1)}(t_\mathrm{k})+\mathrm{m}_\mathrm{k}h^2w_{tt}^{(\mathrm{r}+1)}(t_\mathrm{k})\right), \end{cases} \tag{2.17}$$

and

$$\begin{cases} \left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_0^{(\mathrm{r}+1)}-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_1^{(\mathrm{r}+1)}+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_2^{(\mathrm{r}+1)} \\ -\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_3^{(\mathrm{r}+1)}-\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-3}^{(\mathrm{r}+1)}+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-2}^{(\mathrm{r}+1)}-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}\right. \\ \left.+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-1}^{(\mathrm{r}+1)}+\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_n^{(\mathrm{r}+1)}=\lambda(\alpha_1+\beta_1)[(f_0^{(\mathrm{r})}+f_n^{(\mathrm{r})}) \\ -(q_0^{(\mathrm{r})}\sigma_0+q_n^{(\mathrm{r})}\sigma_n)]+\sum_{\mathrm{k}=n-5}^{\mathrm{k}=n}\left(l_\mathrm{k}w^{(\mathrm{r}+1)}(t_\mathrm{k})+\mathrm{m}_\mathrm{k}h^2w_{tt}^{(\mathrm{r}+1)}(t_\mathrm{k})\right), \end{cases} \tag{2.18}$$

respectively. For non-linear BVPs, system (2.16), (2.17) and (2.18) gives the approximate solution $W_\mathrm{j}^{(\mathrm{r}+1)}$, $\mathrm{j}=1,2,\ldots,n-1$.

3. Truncation error

From (2.16), we have

$$\begin{cases} T_\mathrm{j}^{(\mathrm{r})}(h)=\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathrm{r}+1)}(t_0)-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathrm{r}+1)}(t_1) \\ \quad+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathrm{r}+1)}(t_2)-\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathrm{r}+1)}(t_3)-\Big[\alpha_1 \\ \quad+ph^2q^{(\mathrm{r})}(t_{\mathrm{j}-2})\Big]W^{(\mathrm{r}+1)}(t_{\mathrm{j}-2})-\Big[2(\beta_1-\alpha_1)+qh^2q^{(\mathrm{r})}(t_{\mathrm{j}-1})\Big]W^{(\mathrm{r}+1)}(t_{\mathrm{j}-1}) \\ \quad-\Big[(2\alpha_1-4\beta_1)+rh^2q^{(\mathrm{r})}(t_\mathrm{j})\Big]W^{(\mathrm{r}+1)}(t_\mathrm{j})-\Big[2(\beta_1-\alpha_1) \\ \quad+qh^2q^{(\mathrm{r})}(t_{\mathrm{j}+1})\Big]W^{(\mathrm{r}+1)}(t_{\mathrm{j}+1})-\Big[\alpha_1+ph^2q^{(\mathrm{r})}(t_{\mathrm{j}+2})\Big]W^{(\mathrm{r}+1)}(t_{\mathrm{j}+2}) \\ \quad-\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathrm{r}+1)}(t_{n-3})+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathrm{r}+1)}(t_{n-2}) \\ \quad-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathrm{r}+1)}(t_{n-1})+\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathrm{r}+1)}(t_n) \\ \quad+h^2\Big[\mathrm{p}(f^{(\mathrm{r})}(t_{\mathrm{j}+2})+f^{(\mathrm{r})}(t_{\mathrm{j}-2}))+q(f^{(\mathrm{r})}(t_{\mathrm{j}+1})+f^{(\mathrm{r})}(t_{\mathrm{j}-1}))+rf^{(\mathrm{r})}(t_\mathrm{j})\Big] \\ \quad-\lambda(\alpha_1+\beta_1)[(f^{(\mathrm{r})}(t_0)+f^{(\mathrm{r})}(t_n))-(q^{(\mathrm{r})}(t_0)W^{(\mathrm{r}+1)}(t_0) \\ \quad+q^{(\mathrm{r})}W^{(\mathrm{r}+1)}(t_n))],\quad \mathrm{j}=2,3,\ldots,(n-2). \end{cases} \tag{3.1}$$

Substituting $f^{(\mathrm{r})}(t_\mathrm{j})=w_{tt}^{(\mathrm{r}+1)}(t_\mathrm{j})+q^{(\mathrm{r})}(t_\mathrm{j})w^{(\mathrm{r}+1)}(t_\mathrm{j})$ in (3.1), we get

$$
\begin{cases}
T_{\mathrm{j}}^{(\mathrm{r})}(h) = -2(\alpha_2+\beta_2)\lambda\left[\frac{-W^{(\mathrm{r}+1)}(t_0)+3W^{(\mathrm{r}+1)}(t_1)-3W^{(\mathrm{r}+1)}(t_2)+W^{(\mathrm{r}+1)}(t_3)}{h^3}\right] \\
\qquad +2(\alpha_2+\beta_2)\lambda\left[\frac{W^{(\mathrm{r}+1)}(t_n)-W^{(\mathrm{r}+1)}(t_{n-1})+3W^{(\mathrm{r}+1)}(t_{n-2})-W^{(\mathrm{r}+1)}(t_{n-3})}{h^3}\right] \\
\qquad -2(\alpha_1+\beta_1)\lambda\left[\frac{W_1^{(\mathrm{r}+1)}-W_0^{(\mathrm{r}+1)}}{h}\right]+2(\alpha_1+\beta_1)\lambda\left[\frac{W_n^{(\mathrm{r}+1)}-W_{n-1}^{(\mathrm{r}+1)}}{h}\right] \\
\qquad -(\alpha_1+\beta_1)\lambda W_{tt}^{(\mathrm{r}+1)}(t_0)-(\alpha_1+\beta_1)\lambda W_{tt}^{(\mathrm{r}+1)}(t_n) \\
\qquad -\alpha_1(w^{(\mathrm{r}+1)}(t_{\mathrm{j}+2})+w^{(\mathrm{r}+1)}(t_{\mathrm{j}-2}))-2(\beta_1-\alpha_1)(w^{(\mathrm{r}+1)}(t_{\mathrm{j}+1})+w^{(\mathrm{r}+1)}(t_{\mathrm{j}-1})) \\
\qquad -(2\alpha_1-4\beta_1)w^{(\mathrm{r}+1)}(t_{\mathrm{j}})+ph^2w_{tt}^{(\mathrm{r}+1)}(t_{\mathrm{j}+2})+qh^2w_{tt}^{(\mathrm{r}+1)}(t_{\mathrm{j}+1})+rh^2w_{tt}^{(\mathrm{r}+1)}(t_{\mathrm{j}}) \\
\qquad +qh^2w_{tt}^{(\mathrm{r}+1)}(t_{\mathrm{j}+1})+ph^2w_{tt}^{(\mathrm{r}+1)}(t_{\mathrm{j}+2}).
\end{cases}
\tag{3.2}
$$

After further simplification we obtain,

$$
\begin{cases}
T_{\mathrm{j}}^{(\mathrm{r})}(h) = -2(\alpha_2+\beta_2)\lambda\left[W_{ttt}^{(\mathrm{r}+1)}(t_0)+O(h)\right]+2(\alpha_2+\beta_2)\lambda\left[W_{ttt}^{(\mathrm{r}+1)}(t_n)+O(h)\right] \\
\qquad -2(\alpha_1+\beta_1)\lambda\left[W_{t}^{(\mathrm{r}+1)}(t_0)+O(h)\right]+2(\alpha_1+\beta_1)\lambda\left[W_{t}^{(\mathrm{r}+1)}(t_n)+O(h)\right] \\
\qquad -(\alpha_1+\beta_1)\lambda W_{tt}^{(\mathrm{r}+1)}(t_0)-(\alpha_1+\beta_1)\lambda W_{tt}^{(\mathrm{r}+1)}(t_n) \\
\qquad +\left[\frac{1}{6}(7\alpha_1+\beta_1)-(4p+q)\right]h^4w_{tttt}^{(\mathrm{r}+1)}(t_{\mathrm{j}})+\left[\frac{1}{180}(31\alpha_1+\beta_1)\right. \\
\qquad \left.-\frac{1}{12}(16p+q)\right]h^6w_{tttttt}^{(\mathrm{r}+1)}(t_{\mathrm{j}})+\left[\frac{1}{131040}(1611\alpha_1+31\beta_1)\right. \\
\qquad \left.-\frac{1}{360}(4p+q)\right]h^8w_{tttttttt}^{(\mathrm{r}+1)}(t_{\mathrm{j}})+O(h^9).
\end{cases}
\tag{3.3}
$$

We write

$$T_{\mathrm{j}}^{(\mathrm{r})}(h) = T_{\lambda}^{(\mathrm{r})}(h)+T_{*}^{(\mathrm{r})}(h),$$

where

$$
\begin{aligned}
T_{\lambda}^{(\mathrm{r})}(h) = &-2(\alpha_2+\beta_2)\lambda\left[W_{ttt}^{(\mathrm{r}+1)}(t_0)+O(h)\right]+2(\alpha_2+\beta_2)\lambda\left[W_{ttt}^{(\mathrm{r}+1)}(t_n)+O(h)\right] \\
&-2(\alpha_1+\beta_1)\lambda\left[W_{t}^{(\mathrm{r}+1)}(t_0)+O(h)\right]+2(\alpha_1+\beta_1)\lambda\left[W_{t}^{(\mathrm{r}+1)}(t_n)+O(h)\right] \\
&-(\alpha_1+\beta_1)\lambda W_{tt}^{(\mathrm{r}+1)}(t_0)-(\alpha_1+\beta_1)\lambda W_{tt}^{(\mathrm{r}+1)}(t_n),
\end{aligned}
$$

and

$$
\begin{aligned}
T_{*}^{(\mathrm{r})}(h) = &\left[\frac{1}{6}(7\alpha_1+\beta_1)-(4p+q)\right]h^4w_{tttt}^{(\mathrm{r}+1)}(t_{\mathrm{j}})+\left[\frac{1}{180}(31\alpha_1+\beta_1)-\frac{1}{12}(16p+q)\right]h^6w_{tttttt}^{(\mathrm{r}+1)}(t_{\mathrm{j}}) \\
&+\left[\frac{1}{131040}(1611\alpha_1+31\beta_1)-\frac{1}{360}(4p+q)\right]h^8w_{tttttttt}^{(\mathrm{r}+1)}(t_{\mathrm{j}})+O(h^9).
\end{aligned}
$$

4. Class of methods

4.1. Second order method

Choose λ such that $|\lambda| < h^4$. For getting method of second order, unknown coefficients must satisfy conditions:

$(\alpha_1+\beta_1) = \frac{1}{2}$.

$\left[\frac{1}{6}(7\alpha_1+\beta_1)-(4p+q)\right] \neq 0$.

One such set of values are:

$(\alpha_1, \beta_1) = \left(\frac{1}{4}, \frac{1}{4}\right)$ and
$p = 1/4$, $q = 0$, $r = 1/2$.
Also

at $\mathtt{j} = 1$, $\quad \left(l_0, l_1, l_2, l_3, l_4, l_5\right) = \left(0, -1, 2, -1, 0, 0\right)$,
$\quad \left(\mathtt{m}_0, \mathtt{m}_1, \mathtt{m}_2, \mathtt{m}_3, \mathtt{m}_4, \mathtt{m}_5\right) = \left(0, \frac{1}{6}, \frac{4}{6}, \frac{1}{6}, 0, 0\right)$,

and

at $\mathtt{j} = n-1$, $\quad \left(l_n, l_{n-1}, l_{n-2}, l_{n-3}, l_{n-4}, l_{n-5}\right) = \left(0, -1, 2, -1, 0, 0\right)$,
$\quad \left(\mathtt{m}_n, \mathtt{m}_{n-1}, \mathtt{m}_{n-2}, \mathtt{m}_{n-3}, \mathtt{m}_{n-4}, \mathtt{m}_{n-5}\right) = \left(0, \frac{1}{6}, \frac{4}{6}, \frac{1}{6}, 0, 0\right)$.

Since $|\lambda| < h^4$, we have $T_\lambda^{(\mathtt{r})}(h) = O(h^4)$ and $T_*^{(\mathtt{r})}(h) = \frac{-2}{3} h^4 w_{tttt}^{(\mathtt{r}+1)}(t_\mathtt{j}) + O(h^5)$.
Therefore

$$T_\mathtt{j}^{(\mathtt{r})}(h) = O(h^4). \tag{4.1}$$

4.2. **Fourth order method**

Choose λ such that $|\lambda| < h^6$. For getting method of order four, values of unknown coefficients must satisfy conditions:
$(\alpha_1 + \beta_1) = \frac{1}{2}$,
$\left[\frac{1}{6}\left(7\alpha_1 + \beta_1\right) - (4p + q)\right] = 0$,
$\left[\frac{1}{180}\left(31\alpha_1 + \beta_1\right) - \frac{1}{12}(16p + q)\right] \neq 0$.
One such set of values are $(\alpha_1, \beta_1) = \left(\frac{1}{6}, \frac{1}{3}\right)$ and
$p = \frac{1}{120}, q = \frac{26}{120}, r = \frac{66}{120}$.
Also

at $\mathtt{j} = 1$, $\quad \left(l_0, l_1, l_2, l_3, l_4, l_5\right) = \left(0, -1, 2, -1, 0, 0\right)$,
$\quad \left(\mathtt{m}_0, \mathtt{m}_1, \mathtt{m}_2, \mathtt{m}_3, \mathtt{m}_4, \mathtt{m}_5\right) = \left(0, \frac{1}{12}, \frac{10}{12}, \frac{1}{12}, 0, 0\right)$,

and

at $\mathtt{j} = n-1$, $\quad \left(l_n, l_{n-1}, l_{n-2}, l_{n-3}, l_{n-4}, l_{n-5}\right) = \left(0, -1, 2, -1, 0, 0\right)$,
$\quad \left(\mathtt{m}_n, \mathtt{m}_{n-1}, \mathtt{m}_{n-2}, \mathtt{m}_{n-3}, \mathtt{m}_{n-4}, \mathtt{m}_{n-5}\right) = \left(0, \frac{1}{12}, \frac{10}{12}, \frac{1}{12}, 0, 0\right)$.

Since $|\lambda| < h^6$, we have $T_\lambda^{(\mathtt{r})}(h) = O(h^6)$ and $T_*^{(\mathtt{r})}(h) = \frac{7}{5000} h^6 w_{tttt}^{(\mathtt{r}+1)}(t_\mathtt{j}) + O(h^7)$.
Therefore

$$T_\mathtt{j}^{(\mathtt{r})}(h) = O(h^6). \tag{4.2}$$

4.3. **Sixth order method**

Choose λ such that $|\lambda| < h^8$. For getting method of order six, values of unknown coefficients must satisfy conditions:

$(\alpha_1+\beta_1)=\frac{1}{2}$,
$\frac{1}{6}(7\alpha_1+\beta_1)-(4p+q)=0$,
$\frac{1}{180}(31\alpha_1+\beta_1)-\frac{1}{12}(16p+q)=0$,
$\left[\frac{1}{131040}(1611\alpha_1+31\beta_1)-\frac{1}{360}(4p+q)\right]\neq 0$.
The only set of such values are $(\alpha_1,\beta_1)=\left(\frac{1}{12},\frac{5}{12}\right)$ and $p=\frac{1}{360}, q=\frac{56}{360}, r=\frac{246}{360}$.
Also

at $\mathtt{j}=1$, $\quad \big(l_0,l_1,l_2,l_3,l_4,l_5\big)=\big(-4,7,-2,-1,0,0\big)$,
$\big(\mathtt{m}_0,\mathtt{m}_1,\mathtt{m}_2,\mathtt{m}_3,\mathtt{m}_4,\mathtt{m}_5\big)=\big(\frac{71}{240},\frac{43}{12},\frac{7}{8},\frac{1}{3},\frac{-5}{48},\frac{1}{60}\big)$,

and

at $\mathtt{j}=n-1$, $\quad \big(l_n,l_{n-1},l_{n-2},l_{n-3},l_{n-4},l_{n-5}\big)=\big(-4,7,-2,-1,0,0\big)$,
$\big(\mathtt{m}_n,\mathtt{m}_{n-1},\mathtt{m}_{n-2},\mathtt{m}_{n-3},\mathtt{m}_{n-4},\mathtt{m}_{n-5}\big)=\big(\frac{71}{240},\frac{43}{12},\frac{7}{8},\frac{1}{3},\frac{-5}{48},\frac{1}{60}\big)$.

Since $|\lambda|<h^8$, we have $T_\lambda^{(\mathtt{r})}(h)=O(h^8)$ and $T_*^{(\mathtt{r})}(h)=\frac{7}{5000}h^8 w_{tttttttt}^{(\mathtt{r}+1)}(t_\mathtt{j})+O(h^9)$.
Therefore

$$T_\mathtt{j}^{(\mathtt{r})}(h)=O(h^8). \tag{4.3}$$

Remark 3: Since $\alpha_2=\frac{1}{\xi^2}\left(\frac{1}{6}-\alpha_1\right)$ and $\beta_2=\frac{1}{\xi^2}\left(\frac{1}{3}-\beta_1\right)$,
i.e.$(\alpha_2+\beta_2)=\frac{1}{\xi^2}\left(\frac{1}{2}-(\alpha_1+\beta_1)\right)$,
therefore $(\alpha_1+\beta_1)=\frac{1}{2}$ implies $(\alpha_2+\beta_2)=0$.

5. Convergence analysis

The system given in (2.16), (2.17) and (2.18) can be written as

$$M^{(\mathtt{r})}W^{(\mathtt{r}+1)}=d^{(\mathtt{r})}, \tag{5.1}$$

where

$$M^{(\mathtt{r})}=\begin{bmatrix}
M^{(\mathtt{r})}_{1,1} & M^{(\mathtt{r})}_{1,2} & M^{(\mathtt{r})}_{1,3} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & M^{(\mathtt{r})}_{1,n-3} & M^{(\mathtt{r})}_{1,n-2} & M^{(\mathtt{r})}_{1,n-1}\\
M^{(\mathtt{r})}_{2,1} & M^{(\mathtt{r})}_{2,2} & M^{(\mathtt{r})}_{2,3} & M^{(\mathtt{r})}_{2,4} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & M^{(\mathtt{r})}_{2,n-3} & M^{(\mathtt{r})}_{2,n-2} & M^{(\mathtt{r})}_{2,n-1}\\
M^{(\mathtt{r})}_{3,1} & M^{(\mathtt{r})}_{3,2} & M^{(\mathtt{r})}_{3,3} & M^{(\mathtt{r})}_{3,4} & M^{(\mathtt{r})}_{3,5} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & M^{(\mathtt{r})}_{3,n-3} & M^{(\mathtt{r})}_{3,n-2} & M^{(\mathtt{r})}_{3,n-1}\\
M^{(\mathtt{r})}_{4,1} & M^{(\mathtt{r})}_{4,2} & M^{(\mathtt{r})}_{4,3} & M^{(\mathtt{r})}_{4,4} & M^{(\mathtt{r})}_{4,5} & M^{(\mathtt{r})}_{4,6} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & M^{(\mathtt{r})}_{4,n-3} & M^{(\mathtt{r})}_{4,n-2} & M^{(\mathtt{r})}_{4,n-1}\\
M^{(\mathtt{r})}_{5,1} & M^{(\mathtt{r})}_{5,2} & M^{(\mathtt{r})}_{5,3} & M^{(\mathtt{r})}_{5,4} & M^{(\mathtt{r})}_{5,5} & M^{(\mathtt{r})}_{5,6} & M^{(\mathtt{r})}_{5,7} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & M^{(\mathtt{r})}_{5,n-3} & M^{(\mathtt{r})}_{5,n-2} & M^{(\mathtt{r})}_{5,n-1}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
M^{(\mathtt{r})}_{n-5,1} & M^{(\mathtt{r})}_{n-5,2} & M^{(\mathtt{r})}_{n-5,3} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & M^{(\mathtt{r})}_{n-5,n-7} & M^{(\mathtt{r})}_{n-5,n-6} & M^{(\mathtt{r})}_{n-5,n-5} & M^{(\mathtt{r})}_{n-5,n-4} & M^{(\mathtt{r})}_{n-5,n-3} & M^{(\mathtt{r})}_{n-5,n-2} & M^{(\mathtt{r})}_{n-5,n-1}\\
M^{(\mathtt{r})}_{n-4,1} & M^{(\mathtt{r})}_{n-4,2} & M^{(\mathtt{r})}_{n-4,3} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & M^{(\mathtt{r})}_{n-4,n-6} & M^{(\mathtt{r})}_{n-4,n-5} & M^{(\mathtt{r})}_{n-4,n-4} & M^{(\mathtt{r})}_{n-4,n-3} & M^{(\mathtt{r})}_{n-4,n-2} & M^{(\mathtt{r})}_{n-4,n-1}\\
M^{(\mathtt{r})}_{n-3,1} & M^{(\mathtt{r})}_{n-3,2} & M^{(\mathtt{r})}_{n-3,3} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & M^{(\mathtt{r})}_{n-3,n-5} & M^{(\mathtt{r})}_{n-3,n-4} & M^{(\mathtt{r})}_{n-3,n-3} & M^{(\mathtt{r})}_{n-3,n-2} & M^{(\mathtt{r})}_{n-3,n-1}\\
M^{(\mathtt{r})}_{n-2,1} & M^{(\mathtt{r})}_{n-2,2} & M^{(\mathtt{r})}_{n-2,3} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & M^{(\mathtt{r})}_{n-2,n-4} & M^{(\mathtt{r})}_{n-2,n-3} & M^{(\mathtt{r})}_{n-2,n-2} & M^{(\mathtt{r})}_{n-2,n-1}\\
M^{(\mathtt{r})}_{n-1,1} & M^{(\mathtt{r})}_{n-1,2} & M^{(\mathtt{r})}_{n-1,3} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & M^{(\mathtt{r})}_{n-1,n-3} & M^{(\mathtt{r})}_{n-1,n-2} & M^{(\mathtt{r})}_{n-1,n-1}
\end{bmatrix},$$

where $W^{(\mathtt{r}+1)} = (W_1^{(\mathtt{r}+1)}, W_2^{(\mathtt{r}+1)}, \ldots, W_{n-1}^{(\mathtt{r}+1)})^T$, $M^{(\mathtt{r})}$ is coefficient matrix of $W^{(\mathtt{r}+1)}$and $d^{(\mathtt{r})} = (d_1^{(\mathtt{r})}, d_2^{(\mathtt{r})}, \ldots, d_{n-1}^{(\mathtt{r})})^T$.
Let $N^{(\mathtt{r})}(\mathtt{r})$ be the matrix when $\lambda = 0$. Note that,

$$\|M^{(\mathtt{r})} - N^{(\mathtt{r})}\|_\infty = \max_i \sum_{i=1}^{n-1} \|M_{i,\mathtt{j}}^{(\mathtt{r})} - N_{i,\mathtt{j}}^{(\mathtt{r})}\|.$$

Thus we get

$$\|M^{(\mathtt{r})} - N^{(\mathtt{r})}\|_\infty = 2\left|\frac{2(\alpha_1+\beta_1)\lambda}{h} + \frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right| + 2\left|\frac{-6(\alpha_2+\beta_2)\lambda}{h^3}\right| + 2\left|\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right|.$$

Theorem 5.1. *[7] : Let Q_1andQ_2 be any two matrices having matrix norm as $\|.\|$. If the eigen values of Q_1 are given as $\theta_1, \theta_2, \ldots, \theta_n$ and eigenvalues of Q_2 be given as $\mu_1, \mu_2, \ldots, \mu_{\mathtt{n}}$. Then*

$$\max_{\mathtt{j}} |\theta_{\mathtt{j}} - \mu_{\mathtt{j}}| \leq 2^{\frac{2N-1}{N}} N^{\frac{1}{N}} (2P)^{\frac{N-1}{N}} \|Q_1 - Q_2\|^{\frac{1}{N}}, \tag{5.2}$$

where $P = max(\|Q_1\|,\ \|Q_2\|)$.

In our case, we take the matrices $M^{(\mathtt{r})} = Q_1$, $N^{(\mathtt{r})} = Q_2$, $N = n-1$. Using $\|.\|_\infty$ in theorem 5.1, we get

$$\max_{\mathtt{j}} |\theta_{\mathtt{j}} - \mu_{\mathtt{j}}| \leq 2^{\left(\frac{2n-3}{n-1}\right)} (n-1)^{\left(\frac{1}{n-1}\right)} (2P)^{\left(\frac{n-2}{n-1}\right)} \|M^{(\mathtt{r})} - N^{(\mathtt{r})}\|_\infty^{\left(\frac{1}{n-1}\right)}, \tag{5.3}$$

where $P = max(\|M^{(\mathtt{r})}\|_\infty,\ \|N^{(\mathtt{r})}\|_\infty)$ and $M^{(\mathtt{r})}$ and $N^{(\mathtt{r})}$ have eigenvalues $\theta_{\mathtt{j}}$ *and* $\mu_{\mathtt{j}}, \mathtt{j} = 1, 2, \ldots, n-1$ respectively.

For sufficiently small values of h, $N^{(\mathtt{r})}(\mathtt{r})$ becomes irreducible, $N_{i,i}^{(\mathtt{r})} > 0$, $N_{i,\mathtt{j}}^{(\mathtt{r})} \leq 0$, $i \neq \mathtt{j}$ and the row sums give
$R_1^{(\mathtt{r})} = 4 - \frac{43}{12} h^2 q_1^{(\mathtt{r})} - \frac{7}{8} h^2 q_2^{(\mathtt{r})} - \frac{1}{3} h^2 q_3^{(\mathtt{r})} \quad > 0,$

$R_2^{(\mathtt{r})} = \frac{1}{12} - \frac{56}{360} h^2 q_1^{(\mathtt{r})} - \frac{246}{360} h^2 q_2^{(\mathtt{r})} - \frac{56}{360} h^2 q_3^{(\mathtt{r})} - \frac{1}{360} h^2 q_4^{(\mathtt{r})} \quad > 0,$

$R_{\mathtt{j}}^{(\mathtt{r})} = -\frac{1}{360} h^2 q_{i-2}^{(\mathtt{r})} - \frac{56}{360} h^2 q_{i-1}^{(\mathtt{r})} - \frac{246}{360} h^2 q_i^{(\mathtt{r})} - \frac{56}{360} h^2 q_{i+1}^{(\mathtt{r})} - \frac{1}{360} h^2 q_{i+2}^{(\mathtt{r})} \quad > 0,$
where $\mathtt{j} = 3, 4, \ldots n-3$,

$R_{n-2}^{(\mathtt{r})} = \frac{1}{12} - \frac{56}{360} h^2 q_{n-1}^{(\mathtt{r})} - \frac{246}{360} h^2 q_{n-2}^{(\mathtt{r})} - \frac{56}{360} h^2 q_{n-3}^{(\mathtt{r})} - \frac{1}{360} h^2 q_{n-4}^{(\mathtt{r})} \quad > 0,$

$R_{n-1}^{(\mathtt{r})} = 4 - \frac{43}{12} h^2 q_{n-1}^{(\mathtt{r})} - \frac{7}{8} h^2 q_{n-2}^{(\mathtt{r})} - \frac{1}{3} h^2 q_{n-3}^{(\mathtt{r})} \quad > 0.$

Here $N^{(\mathtt{r})}$ is a monotone matrix [20]. Therefore for adequately small values of h, $(N^{(\mathtt{r})})^{-1}$

exist and we get non-zero eigenvalues $\mu_{\mathtt{j}},\ \mathtt{j}=1,2,\ldots n-1$. Thus for these values of h (corresponding to which $N^{(\mathtt{r})}$ is a monotone matrix), λ lies in the region $(-h^8,\ h^8)$. We select λ in such a manner that it must satisfy the following two conditions :
(i) $M^{(\mathtt{r})}$ is invertible matrix, since $\|M^{(\mathtt{r})}-N^{(\mathtt{r})}\|_\infty=2\left|\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right|+2\left|\frac{-6(\alpha_2+\beta_2)\lambda}{h^3}\right|+2\left|\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right|$, and from (5.3) we find that eigenvalues of $M^{(\mathtt{r})}$ are non-zero, whenever λ is sufficiently small.
(ii) Since $N_{\mathtt{j}}^{(\mathtt{r})}>0,\ \mathtt{j}=1,2,\ldots,n-1,$, the row sum corresponding to $M^{(\mathtt{r})}$ is

$$S_{\mathtt{j}}^{(\mathtt{r})}=R_{\mathtt{j}}-\frac{4(\alpha_1+\beta_1)\lambda}{h}-\frac{4(\alpha_2+\beta_2)\lambda}{h^3},\quad \mathtt{j}=1,2,\ldots,n-1, \tag{5.4}$$

when λ is sufficiently small.
When $N^{(\mathtt{r})}$ is monotone (i.e. when h is adequately small) and $M^{(\mathtt{r})}$ invertible and row sum of $M^{(\mathtt{r})}$ is positive (i.e. for sufficiently small $\lambda\in(-h^8,h^8)$).We derive the error bound as follows:

5.1. Error Bound for Sixth order method

The system (2.16), (2.17), and (2.18) with analytic solutions can be written as

$$M^{(\mathtt{r})}\bar{w}^{(\mathtt{r}+1)}=d^{(\mathtt{r})}+\mathtt{T}^{(\mathtt{r})}(h), \tag{5.5}$$

where

$$\bar{w}^{(\mathtt{r}+1)}=(\bar{w}^{(\mathtt{r}+1)}(t_1),\bar{w}^{(\mathtt{r}+1)}(t_2),\ldots,\bar{w}^{(\mathtt{r}+1)}(t_{n-1}))^T,$$

and

$$\mathtt{T}^{(\mathtt{r})}(h)=(\mathtt{T}_1^{(\mathtt{r})}(h),\mathtt{T}_2^{(\mathtt{r})}(h),\ldots,\mathtt{T}_{n-1}^{(\mathtt{r})}(h))^T.$$

Since from (5.1) we have

$$M^{(\mathtt{r})}W^{(\mathtt{r}+1)}=d^{(\mathtt{r})}. \tag{5.6}$$

Using (5.5) and (5.6) we get

$$M^{(\mathtt{r})}(\bar{w}^{(\mathtt{r}+1)}-W^{(\mathtt{r}+1)})=\mathtt{T}^{(\mathtt{r})}(h),$$

that is,

$$M^{(\mathtt{r})}\mathtt{E}^{(\mathtt{r}+1)}=\mathtt{T}^{(\mathtt{r})}(h), \tag{5.7}$$

where $\mathtt{E}^{(\mathtt{r}+1)} = (\mathtt{E}_1^{(\mathtt{r}+1)}, \mathtt{E}_2^{(\mathtt{r}+1)}, \ldots, \mathtt{E}_{n-1}^{(\mathtt{r}+1)})^T$, $\mathtt{E}_\mathtt{j}^{(\mathtt{r}+1)} = w^{(\mathtt{r}+1)}(t_\mathtt{j}) - W_\mathtt{j}^{(\mathtt{r}+1)}$.
Consequently, using (5.7) we obtain

$$\mathtt{E}^{(\mathtt{r}+1)} = (M^{(\mathtt{r})})^{-1}\mathtt{T}^{(\mathtt{r})}(h). \tag{5.8}$$

Using the definition of product of inverse of matrix with the matrix itself, we get

$$\sum_{\mathtt{j}=1}^{n-1} M_{i,\mathtt{j}}^{(\mathtt{r})^{-1}} \; S_\mathtt{j}^{(\mathtt{r})} = 1, \;\; i = 1,2,\ldots,n-1.$$

Hence by (5.4) we get

$$\sum_{\mathtt{j}=1}^{n-1} M_{i,\mathtt{j}}^{(\mathtt{r})^{-1}} \leq \frac{1}{\underset{1\leq \mathtt{j}\leq n-1}{S_\mathtt{j}^{(\mathtt{r})}}} = \frac{1}{C_i^{(\mathtt{r})}h^2}, \tag{5.9}$$

such that $C^{(\mathtt{r})}$ is constant. Using (5.8) and (5.9) we get

$$\mathtt{E}_i^{(\mathtt{r}+1)} = \sum_{\mathtt{j}=1}^{n-1} M_{i,\mathtt{j}}^{(\mathtt{r})^{-1}} \mathtt{T}_\mathtt{j}^{(\mathtt{r})}(h), \quad i = 1,2,\ldots,n-1. \tag{5.10}$$

Substituting (4.3) and (5.9) in (5.10), we get

$$|\mathtt{E}_i^{(\mathtt{r}+1)}| \leq \frac{qh^8}{C_i^{(\mathtt{r})}h^2},$$

where q is a constant.
Hence we obtain

$$\|E\|_\infty = O(h^6),$$

which proves that the proposed scheme is sixth-order convergent. Similar procedure can be used to derive the convergence of second as well as fourth order methods.

6. Numerical experiments

We take adequate number of iterations till the maximum error between the two succeeding iterations satisfy the following tolerance bound:

$$\max_\mathtt{j} |W_\mathtt{j}^{(\mathtt{r}+1)} - W_\mathtt{j}^{(\mathtt{r})}| < TOL,$$

where TOL is convergence tolerance. When the condition is met, we believe $W^{(\mathtt{r}+1)}$ is the approximate value W of the given problem. Here we have considered $TOL = 10^{-15}$.
For each n, $\mathtt{E}_N$ denotes the maximum point-wise error which is determined by

$$\max_\mathtt{j} |w(t_\mathtt{j}) - W_\mathtt{j}|,$$

where $w(t_{\mathtt{j}})$ and $W_{\mathtt{j}}$ are the analytic and approximate solutions respectively at $t = t_{\mathtt{j}}$. Order of convergence of the proposed method is determined as

$$\mathrm{p}^n = log_2\left(\frac{\mathrm{E}^n}{\mathrm{E}^{2n}}\right).$$

6.1. Numerical Schemes for comparison

As we compare the presented method with Numerov's method and second order finite difference method, here we give a brief particulars about these two methods.

6.1.1. Finite-difference method

Consider BVP given in (1.1) and (1.2), let $W^{(\mathtt{r}+1)}$ be the approximate value of $w^{(\mathtt{r}+1)}(t)$. Putting

$$W_{tt}^{(\mathtt{r}+1)(t)} \approx \frac{1}{h^2}\left[W_{\mathtt{j}-1}^{(\mathtt{r}+1)} - 2W_{\mathtt{j}}^{(\mathtt{r}+1)} + W_{\mathtt{j}+1}^{(\mathtt{r}+1)}\right], \tag{6.1}$$

in (1.2) and after simplifying, we get

$$W_{\mathtt{j}-1}^{(\mathtt{r}+1)} + \left[-2 + h^2 q_{\mathtt{j}}^{(\mathtt{r}+1)}\right] W_{\mathtt{j}}^{(\mathtt{r}+1)} + W_{\mathtt{j}+1}^{(\mathtt{r}+1)} = h^2 \mathtt{f}_{\mathtt{j}}^{(\mathtt{r})}, \tag{6.2}$$

for $\mathtt{j} = 1, 2, \ldots n$. Here $W_0 = \sigma_0$ and $W_1 = \sigma_1$.

6.1.2. Numerov's method

For BVP given in (1.1) and (1.2), Numerov's method can be written as

$$W_{\mathtt{j}-1} - 2W_{\mathtt{j}} + W_{\mathtt{j}+1} = \frac{h^2}{12}\left[\mathtt{f}_{\mathtt{j}-1} + 10\mathtt{f}_{\mathtt{j}} + \mathtt{f}_{\mathtt{j}+1}\right], \tag{6.3}$$

where $\mathtt{f}_{\mathtt{j}} = \mathtt{f}(t_{\mathtt{j}}, W_{\mathtt{j}}),\ \mathtt{j} = 0, 1 \ldots n,\ W_0 = \sigma_0$ and $W_1 = \sigma_1$. To get more details about this method, one can refer [12].

Problem 1: Consider the following linear BVP[25, 31]

$$\begin{cases} w_{tt}(t) + w(t) = -1, & 0 < t < 1, \\ w(0) = 0, & w(1) = 0, \end{cases} \tag{6.4}$$

with exact solution $w(t) = cos(t) + \frac{1-cos(1)}{sin(1)} sin(t) - 1$. Approximate results are shown in Table 1 along with results given by Srivastava et al.[31] and Ramadan et al.[25]. λ varies according to the order of method.

Problem 2: Consider the following nonlinear BVP[3]

$$\begin{cases} w_{tt}(t) + exp(-2w(t)) = 0, & 0 < t < 1, \\ w(0) = 0, & w(1) = log(2), \end{cases} \tag{6.5}$$

Table 1: **M.A.E. for problem 1.**

h	**1/8**	**1/16**	**1/32**	**1/64**
Second Order Method $p = 0.04063483994113$, $q = 0.25412730690212$, $r = 0.41047570631347$	1.5516×10^{-03}	2.0410×10^{-04}	3.0770×10^{-05}	5.2534×10^{-06}
p^N	2.9263	2.7296	2.5502	
$(p,q,r) = \left(\frac{1}{4}, 0, \frac{1}{2}\right)$	3.4324×10^{-03}	6.0707×10^{-04}	1.2491×10^{-04}	2.8070×10^{-05}
p^N	2.4992	2.2809	2.1538	
Fourth Order Method $(p,q,r) = \left(\frac{1}{120}, \frac{26}{120}, \frac{66}{120}\right)$	1.9214×10^{-05}	5.8656×10^{-07}	1.7739×10^{-08}	5.2095×10^{-10}
p^N	5.0337	5.0472	5.0896	
$(p,q,r) = \left(\frac{1}{720}, \frac{11}{45}, \frac{183}{360}\right)$	1.9558×10^{-05}	6.0424×10^{-07}	1.8788×10^{-08}	5.8564×10^{-10}
p^N	5.0164	5.0072	5.0036	
Sixth Order Method $(p,q,r) = \left(\frac{1}{360}, \frac{56}{360}, \frac{246}{360}\right)$	2.6594×10^{-07}	2.2124×10^{-09}	1.6972×10^{-11}	1.2678×10^{-13}
p^N	6.9093	7.0262	7.0646	
Srivastava et al.[31]	7.1329×10^{-08}	5.2213×10^{-09}	3.6359×10^{-10}	3.1275×10^{-11}
p^N	3.7720	3.8440	3.5392	
Ramadan et al.[25]	1.7538×10^{-04}	2.1600×10^{-05}	2.6770×10^{-06}	3.3310×10^{-07}
p^N	3.0213	3.0123	3.0065	

with exact solution $w(t) = log(1+t)$. Approximate results are shown in Table 2 along with results given by Balasubramani et al.[3], finite difference method and Mohanty et al.[24].

Table 2: **M.A.E for problem 2.**

h	**1/8**	**1/16**	**1/32**	**1/64**
Second Order Method				
$(p,q,r) = \left(\frac{1}{4}, 0, \frac{1}{2}\right)$	1.9977×10^{-03}	4.5767×10^{-04}	1.1324×10^{-04}	2.8198×10^{-05}
p^N	2.1259	2.0148	2.0058	
$(p,q,r) = \left(\frac{1}{4}, \frac{1}{4}, 0\right)$	2.7119×10^{-03}	6.2781×10^{-04}	1.5566×10^{-04}	3.8770×10^{-05}
p^N	2.1109	2.0119	2.0053	
Fourth Order Method				
$(p,q,r) = \left(\frac{1}{720}, \frac{11}{45}, \frac{183}{360}\right)$	2.6377×10^{-05}	9.0287×10^{-07}	3.0209×10^{-08}	1.0626×10^{-09}
p^N	4.8686	4.9014	4.8292	
Balasubramani et al.[3]				
$(p,q,r) = \left(\frac{1}{120}, \frac{26}{120}, \frac{66}{120}\right)$	3.7039×10^{-06}	1.3093×10^{-07}	4.6024×10^{-09}	1.6823×10^{-10}
p^N	4.8222	4.8303	4.7739	
Sixth Order Method				
$(p,q,r) = \left(\frac{1}{360}, \frac{56}{360}, \frac{246}{360}\right)$	2.4456×10^{-07}	2.1358×10^{-09}	1.6419×10^{-11}	1.1984×10^{-13}
p^N	6.8392	7.0233	7.0980	
Finite difference method	2.2281×10^{-04}	5.6130×10^{-05}	1.4060×10^{-05}	3.5166×10^{-06}
p^N	1.9890	1.9972	1.9993	
Mohanty et al.[24]	1.6424×10^{-05}	1.0481×10^{-06}	6.5976×10^{-08}	3.8966×10^{-09}
p^N	3.9699	3.9896	4.0816	

Problem 3: Consider the following nonlinear BVP[3]

$$\begin{cases} w_{tt}(t) - \frac{(2-t)exp(2w(t))+(1/(t+1))}{3} = 0, & 0 < t < 1, \\ w(0) = 0, \qquad w(1) = log(1/2), \end{cases} \tag{6.6}$$

with exact solution $w(t) = log(1/1+t)$. Approximate results are shown in Table 3 along with results given by Balasubramani et al.[3], finite difference method and Numerov's method.

Table 3: **M.A.E for problem 3.**

h	**1/8**	**1/16**	**1/32**	**1/64**
Second Order Method $(p,q,r)=\left(\frac{1}{4},0,\frac{1}{2}\right)$	1.3688×10^{-03}	4.1286×10^{-04}	1.1600×10^{-04}	3.0846×10^{-05}
p^N	1.7292	1.8314	1.9110	
$(p,q,r)=\left(\frac{1}{4},\frac{1}{4},0\right)$	2.3839×10^{-03}	6.2248×10^{-04}	1.6526×10^{-04}	4.2528×10^{-05}
p^N	1.9372	1.9132	1.9582	
Fourth Order Method $(p,q,r)=\left(\frac{1}{720},\frac{11}{45},\frac{183}{360}\right)$	2.7594×10^{-05}	9.4434×10^{-07}	3.1573×10^{-08}	1.1062×10^{-09}
p^N	4.8689	4.9025	4.8349	
Balasubramani et al.[3] $(p,q,r)=\left(\frac{1}{120},\frac{26}{120},\frac{66}{120}\right)$	3.8662×10^{-06}	1.3680×10^{-07}	4.8082×10^{-09}	1.7524×10^{-10}
p^N	4.8207	4.8304	4.7781	
Sixth Order Method $(p,q,r)=\left(\frac{1}{360},\frac{56}{360},\frac{246}{360}\right)$	1.3851×10^{-07}	1.2157×10^{-09}	6.9262×10^{-12}	1.2062×10^{-13}
p^N	6.8320	7.4555	5.8434	
Finite difference method	2.3261×10^{-04}	5.8573×10^{-05}	1.4670×10^{-05}	3.6702×10^{-06}
p^N	1.9890	1.9974	1.9989	
Numerov′s Method	2.1034×10^{-06}	1.3382×10^{-07}	8.4017×10^{-09}	5.2577×10^{-10}
p^N	3.9743	3.9935	3.9982	

Problem 4: Consider the following nonlinear BVP[3]

$$\begin{cases} w_{tt}(t) - \frac{25t^8 exp(w(t)) - 20t^3}{4+t^5} = 0, & 0 < t < 1, \\ w(0) = -log(4), & w(1) = -log(5), \end{cases} \tag{6.7}$$

with exact solution $w(t) = -log(4+t^5)$. Approximate results are shown in Table 4 along with results given by Balasubramani et al.[3], finite difference method and Numerov's method.

Table 4: **M.A.E for problem 4.**

h	**1/8**	**1/16**	**1/32**	**1/64**
Second Order Method $(p,q,r) = \left(\frac{1}{4}, 0, \frac{1}{2}\right)$	5.5212×10^{-03}	1.2773×10^{-03}	3.3060×10^{-04}	8.4161×10^{-05}
p^N	2.1118	1.9500	1.9738	
$(p,q,r) = \left(\frac{1}{4}, \frac{1}{4}, 0\right)$	9.5912×10^{-03}	2.0448×10^{-03}	5.2840×10^{-04}	1.3576×10^{-04}
p^N	2.2296	1.9523	1.9605	
Fourth Order Method $(p,q,r) = \left(\frac{1}{720}, \frac{11}{45}, \frac{183}{360}\right)$	6.2487×10^{-05}	1.0123×10^{-06}	3.8928×10^{-08}	2.7550×10^{-09}
p^N	5.9477	4.7007	3.8206	
Balasubramani el al.[3] $(p,q,r) = \left(\frac{1}{120}, \frac{26}{120}, \frac{66}{120}\right)$	3.9439×10^{-06}	3.3929×10^{-07}	1.4653×10^{-08}	6.6424×10^{-10}
p^N	3.5391	4.5332	4.4633	
Sixth Order Method $(p,q,r) = \left(\frac{1}{360}, \frac{56}{360}, \frac{246}{360}\right)$	5.1118×10^{-06}	1.2322×10^{-08}	2.1551×10^{-10}	3.1186×10^{-12}
p^N	8.6963	5.8374	6.1107	
Finite difference Method	1.1795×10^{-03}	2.9324×10^{-04}	7.3024×10^{-05}	1.8265×10^{-05}
p^N	2.0080	2.0056	1.9992	
Numerov′s Method	3.0070×10^{-05}	1.8480×10^{-06}	1.1585×10^{-07}	7.2337×10^{-09}
p^N	4.0242	3.9956	4.0014	

7. Conclusion

This study deals with developing second, fourth and sixth order convergent numerical schemes by using fractal non-polynomial spline function. With the help of quasilinearisation technique, the non-linear BVPs is converted into a system of linear BVPs, which in turn are solved by using the proposed schemes. These schemes are used to find approximate solution

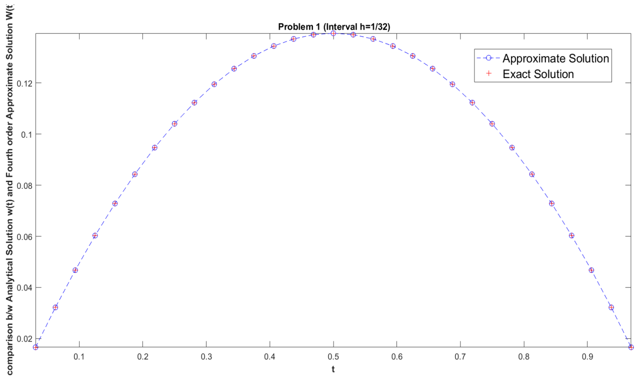

Figure 1: Relationship between analytical and approximate solution for problem 1.

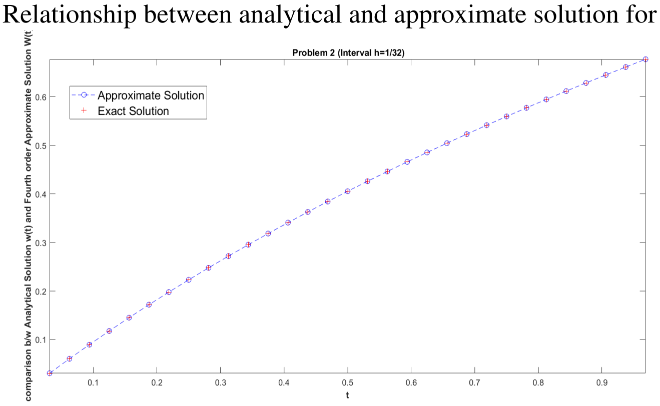

Figure 2: Relationship between analytical and approximate solution for problem 2.

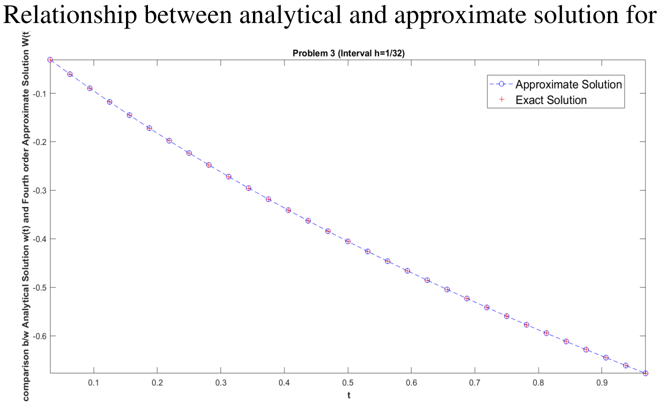

Figure 3: Relationship between analytical and approximate solution for problem 3.

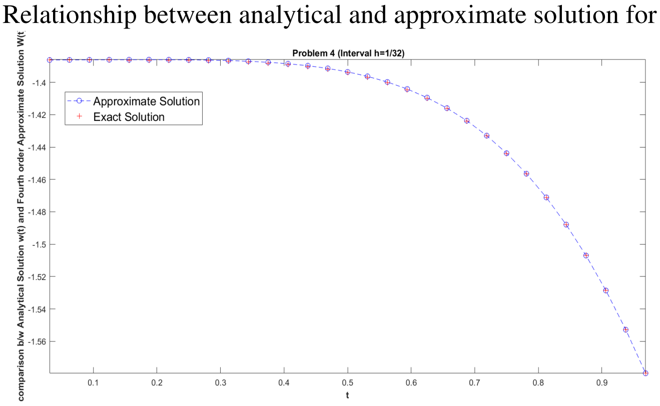

Figure 4: Relationship between analytical and approximate solution for problem 4.

of second order linear as well as nonlinear BVPs. Comparison with polynomial fractal quintic spline and few other methods leads us to the conclusion that the presented methods are more efficient.

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Numerical analysis of Non-Linear Waves Propagation and interactions in Plasma

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Abstract

Solitary wave propagation and interaction in plasma using numerical tools like Galerkin Finite Element scheme are discussed in this paper. A one-dimensional nonlinear Schrodinger-Korteweg De-Vries (Sch-KdV) equation is taken as model equation for Non-linear waves propagation in the said media. The derived system, with the help of cubic B-spline source functions are engaged as element and weight functions, after finite element formulation is solved with Runge Kutta Fourth Order method (RK^4). Previously the finite element methods with some numerical simulations do not exhibit the complex nature of wave interaction, especially solitary wave interaction. A combination of Galerkin Finite Element scheme with RK^4 is a very prominent instrument to study the nature of Non-linear evolution equations in ionic medias, which is the novelty of the paper.

Key words: Schrödinger - Korteweg - De Vries (Sch-KdV)equations, Galerkin Finite Element Scheme,Cubic B-spline source functions, Solitary Wave

Mathematics Subject Classification(2010): 35M10, 65Z05.

1

1 Introduction

Several physical phenomena are described either by nonlinear coupled partial differential equations or by nonlinear evolution equation. This Non-linear wave propagation phenomenon appears in one or other ways can be well explained by travelling and solitary wave solution of the said equations. Most of these equations do not have an analytical solution, or it is extremely difficult and expensive to compute their analytical solutions. Hence numerical study of these nonlinear partial differential equations is important in practice. The Non-linear

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waves propagation in plasma can also be explained by these solutions. In the past study, many methods for finding the Solitary and periodic solutions [1]-[8] and numerical method [8]-[12],[21]-[24] are used for Non-linear evolution equations (NLEEs).
In this paper, we study a Galerkin finite element Scheme for the 1D nonlinear Schrödinger -Korteweg-De -Vries (Sch-KdV) equation by using linear finite elements in space and extrapolation to remove the nonlinear term. We discuss the properties of this method and compare its accuracy with previous studies. The interaction of two solitons is also studied. Moreover, the propagation of the Maxwellian initial condition is simulated.
The outline of this paper is as follows, In the next section the model equation is discretized to form a numerical scheme. In section 3 a numerical scheme is developed and results are explained graphically. Finally, we give a brief conclusion in Section 4

2 Model Equation and Discretization

Non-linear waves propagation and interactions in plasma for this purpose we consider the 1D nonlinear Schrödinger -Korteweg-De -Vries (Sch-KdV) equation [13]-[15] as model equation as -

$$i\theta_t = \theta_{xx} + \theta\upsilon \tag{2.1}$$

$$\upsilon_t = -6\theta\upsilon_x - \upsilon_{xxx} + (|\theta|^2)_x \tag{2.2}$$

Here $\theta(x,t)$is complex function and υ(x,t)is real-valued function. This system appeared as model equation for describing various types of wave propagation such as Langmuir wave, dust-acoustic wave and electromagnetic waves in plasma physics. with initial conditions

$$\theta(x,0) = f(x) = 9\sqrt{2}\, e^{i\alpha x}\, k^2 \text{sech}^2(kx) \tag{2.3}$$

$$\upsilon(x,0) = g(x) = \frac{\alpha + 16k^2}{3} - 6k^2 \tanh^2(kx) \tag{2.4}$$

and boundary conditions

$$\theta(t,a) = 0,\ \upsilon(t,b) = 0,\ \ x \in [a,b]\, and\ t \in [0,T] \tag{2.5}$$

Here $\theta = \theta(x,t)$ and $\upsilon = \upsilon$(x,t) are going to be considered as sufficiently differentiable functions.
We multiplied weight function to the equations (2.1)-(2.2) and integrated over the x domain for finite element method [16]-[20], so we get

$$\int_a^b (i\omega\theta_t - \omega\theta_{xx} - \omega\theta\upsilon)dx = 0 \tag{2.6}$$

$$\int_a^b (\omega\upsilon_t + 6\omega\theta\upsilon_x + \omega\upsilon_{xxx} - \omega(|\theta|^2)_x)dx = 0 \tag{2.7}$$

The domain [a, b] of x is separated into N finite subdivision as

2

$$a = x_0 < x_1 < x_2 < ... < x_{N-1} < x_N = b$$

Here nodal point is $\{x_m\}_{m=0}^{N}$ i.e. m = 0,1,2,...,N and length of subdivision will be $h = x_{m+1}$ - x_m . We construct the approximate solutions for the system with cubic B-spline base functions

$$\theta_N(x,t) = \sum_{j=-1}^{N+1} \psi_j(x) u_j(t) \tag{2.8}$$

$$\upsilon_N(x,t) = \sum_{j=-1}^{N+1} \psi_j(x) v_j(t) \tag{2.9}$$

where u_j(t) and v_j(t) are function of time t and ψ_j(x) are function of x, called element size functions. A local coordinate $\xi = x - x_m$ for 0$\leq \xi \leq$ h introduced for cubic B-spline base functions with typical element $[x_m, x_{m+1}]$, which has the form;

$$\begin{aligned}
\psi_{m-1} &= \frac{(h-\xi)^3}{h^3} \\
\psi_m &= \frac{(h^3 + 3h^2(h-\xi) + 3h(h-\xi)^2 - 3(h-\xi)^3)}{h^3} \\
\psi_{m+1} &= \frac{(h^3 + 3h^2\xi + 3h\xi^2 - 3\xi^3)}{h^3} \\
\psi_{m+2} &= \frac{\xi^3}{h^3}
\end{aligned} \tag{2.10}$$

The approximate solutions of Eqs.((2.8)-(2.9)) with element size function eq.(2.10) may be define as with typical element $[x_m, x_{m+1}]$;

$$\theta_N(\xi,t) = \sum_{j=m-1}^{m+2} u_j^e(t) \psi_j^e(\xi) \tag{2.11}$$

$$\upsilon_N(\xi,t) = \sum_{j=m-1}^{m+2} v_j^e(t) \psi_j^e(\xi) \tag{2.12}$$

The point-wise values of θ_N and υ_N in terms u and v will be

$$\theta_N(x_m,t) = u_{m-1} + 4u_m + u_{m+1} \tag{2.13}$$

$$\upsilon_N(x_m,t) = v_{m-1} + 4v_m + v_{m+1} \tag{2.14}$$

So Eqs. ((2.6)-(2.7)) with $[x_m, x_{m+1}]$ will be

$$\int_{x_m}^{x_{m+1}} (i\omega\theta_t - \omega\theta_{xx} - \omega\theta\upsilon)dx \tag{2.15}$$

$$\int_{x_m}^{x_{m+1}} (\omega\upsilon_t + 6\omega\theta\upsilon_x + \omega_{xx}\upsilon_x - 2\omega\theta\theta_x)dx + [\omega\upsilon_{xx} - \omega_x\upsilon_x] \tag{2.16}$$

here weight function ω_i with size functions ψ_j are takenfor the Galerkin finite element method, Substituting Eqs. ((2.11)-(2.12)) into Eqs. ((2.15)-(2.16)),we get

$$\sum_{j=m-1}^{m+2}\{\int_0^h[(i\psi_i\psi_j)\dot{u}_j-(\psi_i\psi_j^{''})u_j-\sum_{k=m-1}^{m+2}((\psi_i\psi_j\psi_k)u_j)u_k]dx\}=0 \qquad (2.17)$$

$$\sum_{j=m-1}^{m+2}\{\int_0^h[(\psi_i\psi_j)\dot{v}_j+(\psi_j^{"}\psi_k')v_j+\sum_{k=m-1}^{m+2}((6(\psi_i\psi_j\psi_k')u_j)v_k-2((\psi_i\psi_j\psi_k')u_j)u_k)]dx+[((\psi_i\psi_j^{"})-(\psi_i'\psi_j'))v_j]_0^h\}=0 \qquad (2.18)$$

where i, j, k = m-1, m, m+1, m+2, $u^e=(u_{m-1},\ u_m,\ u_{m+1},\ u_{m+2})$ and $v^e=(v_{m-1},\ v_m, v_{m+1},\ u_{m+2})$ are element parameters where

$$A_{ij}=\int_0^h(i\psi_i\psi_j)d\xi,\ \ B_{ij}=\int_0^h(\psi_i\psi_j^{''})d\xi,\ \ C_{jk}=\int_0^h(\psi_j^{"}\psi_k')d\xi$$

$$D_{ij}=\int_0^h(\psi_i\psi_j)d\xi,\ F_{ijk}=\int_0^h 6(\psi_i\psi_j\psi_k')d\xi,\ G_{ijk}=\int_0^h(\psi_i\psi_j\psi_k)d\xi$$

$$H_{ijk}=\int_0^h 2(\psi_i\psi_j\psi_k')d\xi,\qquad I_{ij}=[(\psi_i\psi_j^{"})]_0^h,\quad J_{ij}=[(\psi_i'\psi_j^{'})]_0^h$$

The element matrices in ((2.17)-(2.18)) are computed as follows:

$$A_{ij}=\frac{ih}{140}\begin{bmatrix}20&129&60&1\\129&1188&933&60\\60&933&1188&129\\1&60&129&20\end{bmatrix}\qquad B_{ij}=\frac{3}{10h}\begin{bmatrix}4&-7&2&1\\33&-44&-11&22\\22&-11&-44&33\\1&2&-7&4\end{bmatrix}$$

$$C_{ij}=\frac{3}{2h^2}\begin{bmatrix}-3&-5&7&1\\5&3&-9&1\\-1&9&-3&-5\\-1&-7&5&3\end{bmatrix}\qquad D_{ij}=\frac{h}{140}\begin{bmatrix}20&129&60&1\\129&1188&933&60\\60&933&1188&129\\1&60&129&20\end{bmatrix}$$

$$I_{ij}=\frac{6}{h^2}\begin{bmatrix}-1&2&-1&0\\-4&9&-6&1\\-1&6&-9&4\\0&1&-2&1\end{bmatrix}\qquad J_{ij}=\frac{9}{h^2}\begin{bmatrix}-1&0&1&0\\0&1&0&-1\\1&0&-1&0\\0&-1&0&1\end{bmatrix}$$

$$G_{ij}(u)=\frac{h}{840}\begin{bmatrix}G_{11}(u)&G_{12}(u)&G_{13}(u)&G_{14}(u)\\G_{21}(u)&G_{22}(u)&G_{23}(u)&G_{24}(u)\\G_{31}(u)&G_{32}(u)&G_{33}(u)&G_{34}(u)\\G_{41}(u)&G_{42}(u)&G_{43}(u)&G_{44}(u)\end{bmatrix}$$

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$$F_{ij}(v) = \frac{6h}{840}\begin{bmatrix} F_{11}(v) & F_{12}(v) & F_{13}(v) & F_{14}(v) \\ F_{21}(v) & F_{22}(v) & F_{23}(v) & F_{24}(v) \\ F_{31}(v) & F_{32}(v) & F_{33}(v) & F_{34}(v) \\ F_{41}(v) & F_{42}(v) & F_{43}(v) & F_{44}(v) \end{bmatrix};$$

$$H_{ij}(u) = \frac{2h}{840}\begin{bmatrix} H_{11}(u) & H_{12}(u) & H_{13}(u) & H_{14}(u) \\ H_{21}(u) & H_{22}(u) & H_{23}(u) & H_{24}(u) \\ H_{31}(u) & H_{32}(u) & H_{33}(u) & H_{34}(u) \\ H_{41}(u) & H_{42}(u) & H_{43}(u) & H_{44}(u) \end{bmatrix}$$

where
$G_{11}(u)$=(84,463,172,1)(u), $G_{12}(u)$=(463,2889,1275,17)(u),
$G_{13}(u)$=(172,1275,696,17)(u), $G_{14}(u)$=(1,17,17,1)(u)
$G_{21}(u)$=(463,2889,1275,17)(u), $G_{22}(u)$=(2889,23664,15519,696)(u)
$G_{23}(u)$=(1275,15519,15519,1275)(u), $G_{24}(u)$=(17,696,1275,172)(u),
$G_{31}(u)$=(172,1275,696,17)(u), $G_{32}(u)$=(1275,15519,15519,1275)(u),
$G_{33}(u)$=(696,15519,23664,2889)(u), $G_{34}(u)$=(17,1275,2889,463)(u),
$G_{41}(u)$=(1,17,17,1)(u), $G_{42}(u)$=(17,696,1275,172)(u),
$G_{43}(u)$=(17,1275,2889,463)(u), $G_{44}(u)$=(1,172,463,84)(u)

$F_{11}(v)$=(-280,-150,420,10)(v) $F_{12}(v)$=(-1605,-1305,2781,129)(v)
$F_{13}(v)$=(-630,-792,1314,108)(v) $F_{14}(v)$=(-5,-21,21,5)(v)
$F_{21}(v)$=(-1605,-1305,2781,129)(v) $F_{22}(v)$=(-10830,-17640,25002,3468)(v)
$F_{23}(v)$=(-5349,-17541,17541,5349)(v) $F_{24}(v)$=(-108,-1314,792,630)(v)
$F_{31}(v)$=(-630,-792,1314,108)(v) $F_{32}(v)$=(-5349,-17541,17541,5349)(v)
$F_{33}(v)$=(-3468,-25002,17640,10830)(v) $F_{34}(v)$=(-129,-2781,1305,1605)(v)
$F_{41}(v)$=(-5,-21,21,5)(v) $F_{42}(v)$=(-108,-1314,792,630)(v)
$F_{43}(v)$=(-129,-2781,1305,1605)(v) $F_{44}(v)$=(-10,-420,150,280)(v)

$H_{11}(u)$=(-280,-150,420,10)(u) $H_{12}(u)$=(-1605,-1305,2781,129)(u)
$H_{13}(u)$=(-630,-792,1314,108)(u) $H_{14}(u)$=(-5,-21,21,5)(u)
$H_{21}(u)$=(-1605,-1305,2781,129)(u) $H_{22}(u)$=(-10830,-17640,25002,3468)(u)
$H_{23}(u)$=(-5349,-17541,17541,5349)(u) $H_{24}(u)$=(-108,-1314,792,630)(u)
$H_{31}(u)$=(-630,-792,1314,108)(u) $H_{32}(u)$=(-5349,-17541,17541,5349)(u)
$H_{33}(u)$=(-3468,-25002,17640,10830)(u) $H_{34}(u)$=(-129,-2781,1305,1605)(u)
$H_{41}(u)$=(-5,-21,21,5)(u) $H_{42}(u)$=(-108,-1314,792,630)(u)
$H_{43}(u)$=(129,2781,1305,1605)(u) $H_{44}(u)$=(-10,-420,150,280)(u)

Here A_{ij}, B_{ij}, C_{jk}, D_{ij}, F_{ijk}, G_{ijk}, H_{ijk}, I_{ij} and J_{ij} are element matrices. so, the new obtained system in matrix form

$$\dot{u} = A^{-1}[\{B - G(u)\}u] \tag{2.19}$$

$$\dot{v} = D^{-1}[H(u)u + (J - I)v - Cv - F(u)v] \tag{2.20}$$

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Here $u=(u_{-1},u_0,u_1,...,u_N,u_{N+1})$ and $v=(v_{-1},v_0,v_1,...,v_N,v_{N+1})$ are time dependent constraints, The generalized rows of the combined matrices are:

A = $\frac{ih}{140}$(1,120,1191,2416,1191,120,1)
B = $\frac{3}{10h}$(1,24,15,-80,15,24,1)
C = $\frac{3}{2h^2}$(-1,-8,19,0,-19,8,1)
D = $\frac{h}{140}$(1,120,1191,2416,1191,120,1)
I = $\frac{6}{h^2}$(0,0,0,0,0,0,0)
J = $\frac{9}{h^2}$ (0,0,0,0,0,0,0)
G(u) = $\frac{h}{840}$$\{$(1,17,17,1,0,0,0)u,(17,868,2550,868,17,0,0)u,(17,2550,18871,18871,2550,17,0)u, (1,868,18871,47496,18871,868,1)u,(0,17,2550,18871,18871,2550,17)u,(0,0,17,868,2550,868, 17)u,(0,0,0,1,17,17,1)u$\}$

F(v) = $\frac{6h}{840}$ $\{$(-5,-21,21,5,0,0,0)v,(-108,-1944,0,1944,108,0,0)v,(-129,-8130,-17841,17841,8130, 129,0)v, (-10,-3888,-35682,0,35682,3888,10)v,(0,-129,-8130,-17841,17841,8130,129)v,(0,0,-108,-1944,0,1944,108)v,
(0,0,0,-5,-21,21,5)v$\}$

H(u) = $\frac{2h}{840}$ $\{$(-5,-21,21,5,0,0,0)u,(-108,-1944,0,1944,108,0,0)u,(-129,-8130,-17841,17841,8130, 129,0)u, (-10,-3888,-35682,0,35682,3888,10)u,(0,-129,-8130,-17841,17841,8130,129)u,(0,0,-108,-1944,0,1944,108)u, (0,0,0,-5,-21,21,5)u $\}$

The system equations (2.19) and (2.20) has $(N+3)\times(N+1)$ ordered unknown equations. if we use time dependent boundary condition in Eqs.(2.13) and (2.14) with $m=0$, then so parameters can be written as other parameters;

$$u_{-1},\ v_{-1}\rightarrow u_0,\ u_1 \text{ and } v_0,v_1 \text{ ; when we take } m=0$$

Similarly

$$u_{N+1}\ ,\ v_{N+1}\rightarrow u_{N-1},u_N \text{ and } v_{N-1},v_N \text{ we take } m=N$$

Then, the system of Eqs. (2.19) and(2.20) will be two matrix systems of $(N+1)\times(N+1)$ orders.These equations of systems will be solved by RK^4 (Runge-Kutta fourth order method) to known initial condition u_j^0 and v_j^0 with nodal points x_m for m=0(1)N as follows:

$$u(x_m,0)=\theta_N(x_m,0)$$

$$v(x_m,0)=v_N(x_m,0)$$

If we write the system explicitly as

$$\theta_N(x_0,0)=u_{-1}+4u_0+u_1=u(x_0,0),$$

$$\theta_N(x_1,0)=u_0+4u_1+u_2=u(x_1,0),$$

$$\theta_N(x_2,0)=u_1+4u_2+u_3=u(x_2,0),$$

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$$\vdots$$

$$\theta_N(x_N, 0) = u_{N-1} + 4u_N + u_{N+1} = u(x_N, 0),$$

and

$$\upsilon_N(x_0, 0) = \upsilon_{-1} + 4\upsilon_0 + \upsilon_1 = \upsilon(x_0, 0),$$
$$\upsilon_N(x_1, 0) = \upsilon_0 + 4\upsilon_1 + \upsilon_2 = \upsilon(x_1, 0),$$
$$\upsilon_N(x_2, 0) = \upsilon_1 + 4\upsilon_2 + \upsilon_3 = \upsilon(x_2, 0),$$
$$\vdots$$
$$\upsilon_N(x_N, 0) = \upsilon_{N-11} + 4\upsilon_N + \upsilon_{N+1} = \upsilon(x_N, 0),$$

if we write u_{-1} , $u_{N+1} \to u_0$, u_N, and υ_{-1} , $\upsilon_{N+1}\to$ υ_0 and υ_N respectively. then we get a new system $(N + 1) \times (N + 1)$ order in matrix form as :

$$\begin{bmatrix} 4 & 2 & & & & & & \\ 1 & 4 & 1 & & & & & \\ & 1 & 4 & 1 & & & & \\ & & & . & & & & \\ & & & & . & & & \\ & & & & & . & & \\ & & & & & 1 & 4 & 1 \\ & & & & & & 2 & 4 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ . \\ . \\ . \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} u(x_0, 0) \\ u(x_1, 0) \\ u(x_2, 0) \\ . \\ . \\ . \\ u(x_{N-1}, 0) \\ u(x_N, 0) \end{bmatrix} \tag{2.21}$$

and

$$\begin{bmatrix} 4 & 2 & & & & & & \\ 1 & 4 & 1 & & & & & \\ & 1 & 4 & 1 & & & & \\ & & & . & & & & \\ & & & & . & & & \\ & & & & & . & & \\ & & & & & 1 & 4 & 1 \\ & & & & & & 2 & 4 \end{bmatrix} \begin{bmatrix} \upsilon_0 \\ \upsilon_1 \\ \upsilon_2 \\ . \\ . \\ . \\ \upsilon_{N-1} \\ \upsilon_N \end{bmatrix} = \begin{bmatrix} \upsilon(x_0, 0) \\ \upsilon(x_1, 0) \\ \upsilon(x_2, 0) \\ \upsilon(x_{N-1}, 0) \\ \upsilon(x_N, 0) \end{bmatrix} \tag{2.22}$$

By Matleb solving the algebraic Equations (2.21) and (2.22) with initial parameters u_j^0 and υ_j^0 are gained for j=0(1)N.

3 Numerical Scheme

Non-Linear waves propagations and interaction are investigated to the system of equations (2.1)-(2.2) numerically for numerous values of x and t. L_2 , L_∞ and L'_2 , L'_∞ are error norms and used to investigate consistency with numerical solutions(Soliton) for $\theta(x, t)$ and $\upsilon(x, t)$ respectively for initial conditions for the Sch-KdV equation.

$$\theta(x, 0) = f(x) = 9\sqrt{2}\, e^{i\alpha x}\, k^2\text{sech}^2(kx), \tag{3.1}$$

$$\upsilon(x,0) = g(x) = \frac{\alpha + 16k^2}{3} - 6k^2 \tanh^2(kx) \tag{3.2}$$

$$L_2 = \|\theta - \theta_N\|_2 = \sqrt{h \sum_{j=-1}^{N+1} \left|\theta_j - (\theta_N)_j\right|^2} \tag{3.3}$$

$$L_\infty = \|\theta - \theta_N\|_\infty = \underset{0 \le j \ge N}{Max} \left|\theta_j - (\theta_N)_j\right| \tag{3.4}$$

And

$$L'_2 = \|\upsilon - \upsilon_N\|_2 = \sqrt{h \sum_{j=-1}^{N+1} \left|\upsilon_j - (\upsilon_N)_j\right|^2} \tag{3.5}$$

$$L'_\infty = \|\upsilon - \upsilon_N\|_\infty = \underset{0 \le j \ge N}{Max} \left|\upsilon_j - (\upsilon_N)_j\right| \tag{3.6}$$

Numerical error L_2 *and* L_∞ For $\theta(x,t)$ with $k = \sqrt{2},\ \alpha = 1/20$

h	$\Delta t = 0.001$	$\Delta t = 0.002$	$\Delta t = 0.003$	$\Delta t = 0.01$
	L_2 ; L_∞	L_2 ; L_∞	L_2 ; L_∞	L_2 ; L_∞
0.2	15.82×10^{-7} ; 51.24×10^{-7}	30.71×10^{-7} ; 95.17×10^{-7}	41.39×10^{-7} ; 97.41×10^{-7}	55.56×10^{-7} ; 99.56×10^{-7}
0.4	54.21×10^{-7} ; 55.88×10^{-7}	63.56×10^{-7} ; 68.67×10^{-7}	71.39×10^{-7} ; 70.70×10^{-7}	86.66×10^{-7} ; 85.01×10^{-7}
0.625	57.24×10^{-7} ; 59.82×10^{-7}	68.21×10^{-7} ; 61.23×10^{-7}	78.29×10^{-7} ; 68.21×10^{-7}	87.21×10^{-7} ; 82.01×10^{-7}
0.8	68.19×10^{-7} ; 64.21×10^{-7}	72.21×10^{-7} ; 59.52×10^{-7}	80.19×10^{-7} ; 63.11×10^{-7}	89.21×10^{-7} ; 78.21×10^{-7}
0.1	75.21×10^{-7} ; 72.24×10^{-7}	75.11×10^{-7} ; 52.11×10^{-7}	83.12×10^{-7} ; 59.29×10^{-7}	91.11×10^{-7} ; 72.31×10^{-7}

Numerical error L_2 *and* L_∞ For $\upsilon(x,t)$

h	$\Delta t = 0.001$	$\Delta t = 0.002$	$\Delta t = 0.003$	$\Delta t = 0.01$
	L'_2 ; L'_∞	L'_2 ; L'_∞	L'_2 ; L'_∞	L'_2 ; L'_∞
0.25	07.85×10^{-8} ; 05.68×10^{-8}	08.75×10^{-7} ; 07.05×10^{-7}	15.16×10^{-7} ; 06.65×10^{-7}	19.56×10^{-7} ; 08.75×10^{-7}
0.5	09.95×10^{-8} ; 06.95×10^{-8}	10.72×10^{-7} ; 08.75×10^{-7}	20.61×10^{-7} ; 08.85×10^{-7}	25.11×10^{-7} ; 09.85×10^{-7}
0.625	10.01×10^{-8} ; 07.02×10^{-8}	13.56×10^{-7} ; 09.11×10^{-7}	22.56×10^{-7} ; 09.21×10^{-7}	26.92×10^{-7} ; 10.96×10^{-7}
0.8	12.21×10^{-8} ; 08.11×10^{-8}	15.11×10^{-7} ; 10.21×10^{-7}	24.11×10^{-7} ; 10.09×10^{-7}	28.11×10^{-7} ; 12.11×10^{-7}
0.1	14.02×10^{-8} ; 09.75×10^{-8}	19.21×10^{-7} ; 11.25×10^{-7}	27.72×10^{-7} ; 12.21×10^{-7}	31.27×10^{-7} ; 14.25×10^{-7}

In figure 1 and 2 nonlinear wave propagation and its travelling wave solution is presented. The coupled equations (2.1) and (2.2) are plotted for some fix values of k, α,h and t $(-5 < t < 5)$. the space step is taken as 0.001. It is shown in the figure that the solution of said equation exhibit a soliton for the small values of x $(0 \le x \le 0.1)$. If we extend the range of x $(-15 \le x \le 15)$ the solution converted from soliton to a wave natured system. A solitary wave interaction is presented in the figure 3 for the same values of k, α, h and step lengths with

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Solitary wave propagation for model equations

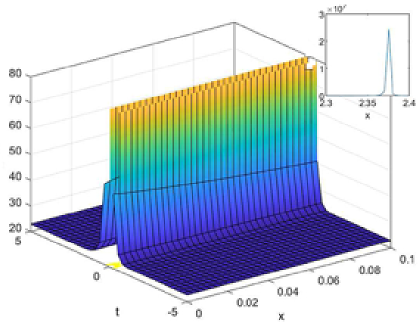

Figure 1. Modulus in 3D, 2D plot the *solitary wave propagation of* θ *when* $k = \sqrt{2},\ \alpha = 1/20, h = 0.4$ $\Delta t = 0.001,\ \Delta x = 0.001, -5 \le t \le 5,\ 0 \le x \le 0.1,$

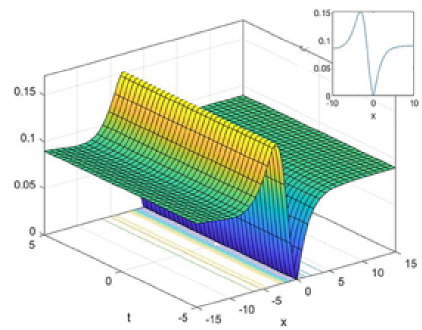

Figure 2. Modulus in 3D, 2D plot the *solitary wave propagation of* υ *when* $k = \sqrt{2},\ \alpha = 1/20, h = 0.4$ $\Delta t = 0.001,\ \Delta x = 0.001, -5 \le t \le 5,\ -15 \le x \le 15,$

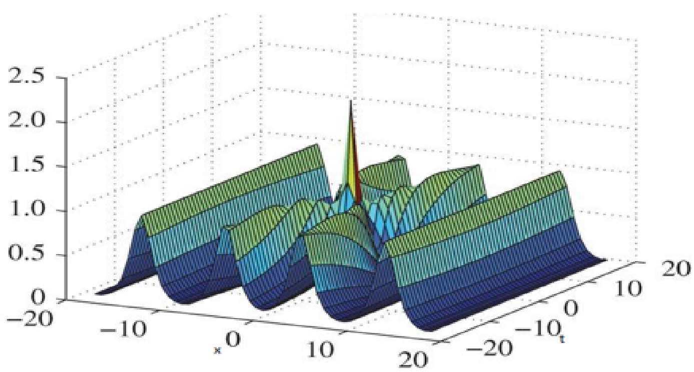

Fig.-3 Solitary wave interaction
$k = \sqrt{2},\ \alpha = 1/20,\ \Delta t = 0.02,\ \Delta x = 0.02,\ -20 \le (x,t) \le 20,$

large values of x and t $(-20 \le (x,t) \le 20)$. It clearly exhibit that solitons are developed when the values of x and t coincides. For different values of x and t the system represent the travelling wave solution.

4 Conclusions

In the present paper, we have investigated numerically a physical model for wave propagation in a nonlinear, dispersive medium i.e a relativistic plasma. A Galerkin finite element Scheme is exhibited to locate Solitary wave(Solitons) propagation and interactions in plasma for Schrödinger - KortewegDe Vries (Sch-KdV) equations. The new obtained systems (finite element formulation) solved

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by RK^4 (Runge-Kutta fourth order method). The different values of x, t and error norms L_2 , L_∞ are used for numerical solutions of Sch-KdV equations.The numerical results obtained by this method are in good agreement with the exact solutions available in the literature. The errors obtained by the proposed method are less when compared with those of available in the literature. The solitary wave solution in fig.-1, 2 and its interaction in fig.-3 of this system are presented which are new. here, we learn that this method will emulates development of many exact travelling wave solutions with new solitons.This scheme is a significant instrument for Non-linear evolution equations (NLEEs).
The advantages of the present scheme for oscillatory problems are discussed in detail. It can be expected that the main ideas will also be useful for other physical problems being highly oscillatory in nature, e.g., the nonlinearized model.

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MULTIPLE SUMMATION FORMULAE FOR THE MODIFIED MULTIVARIABLE I-FUNCTION

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ABSTRACT. The importance of I-function, H-function and many more special functions has a wide range of applications in applied mathematics and applied physics. Some of the multiple summations for the modified multivariable I-function(MMIF) has been discussed in the present article. Some of the summation formulae are concluded at the end of the paper as special cases of our primary results. Also these summation formulae leads to develop the solution of a boundary value problem.

1. INTRODUCTION

Recent advancements of special functions and their applications in mathematical modelling attracting researchers. The motivation of this work is by the applications of special functions like G, H and I-functions by several authors([1], [2], [3]). The generalization of H-function, namely I-function has great importance in Physics and Applied Mathematics. Prasad [15] generalized the I-function and studied many results. In the literature of the special functions like H, G, Meijer etc., many authors established integral results and solved boundary value problems also([7], [11], [5]). Recently, I-function has found its applications in wireless communication.

Srivastava and Panda [8, 9] studied multivariable H-function. The extension of the same as two functions H and I studied by Prasad and Singh [14, 15]. Here we establish four different summation formulae for the MMIF defined by Prasad [15] and a number of summation formulae derived as particular cases.

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Assume $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ as set of complex, real and positive integers respectively and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We define MMIF as :

(1.1) $$I(Z_1, ..., Z_r) =$$

$$I^{0,n_2;0,n_3;....;0,n_r:\left|R^1:m^1,n^1;....;m^{(r)},n^{(r)}\right.}_{p_2,q_2;p_3,q_3;....;p_r,n_r:\left|R:p^1,q^1;....;p^{(r)},q^{(r)}\right.} \left[\begin{matrix} Z_1 \\ \vdots \\ Z_r \end{matrix} \left| \begin{matrix} \left(a_{2j};\alpha^1_{2j},\alpha^{11}_{2j}\right)_{1,p_2};\left(\alpha_{3j};\alpha^1_{3j},\alpha^{11}_{3j},\alpha^{111}_{3j}\right)_{1,p_3}; \\ \\ \left(b_{2j};\beta^1_{2j},\beta^{11}_{2j}\right)_{1,q_2};\left(\beta_{3j};\beta^1_{3j},\beta^{11}_{3j},\beta^{111}_{3j}\right)_{1,q_3}; \end{matrix} \right. \right.$$

$$...; (a_{rj};\alpha'_{rj},...,\alpha^{(r)}_{rj})_{1,p_r} : (e_j; u'_j g'_j, ..., u^{(r)}_j g^{(r)}_j)_{1,R'} :$$

$$\vdots$$

$$...; (b_{rj};\beta'_{rj},...,\beta^{(r)}_{rj})_{1,q_r} : (l_j; U'_j f'_j, ..., U^{(r)}_j f^{(r)}_j)_{1,R} :$$

$$\left. \begin{matrix} (a'_j;\alpha'_j)_{1,p^{(1)}}, (a^{(r)}_j;\alpha^{(r)}_j)_{1,p^{(r)}} \\ \vdots \\ (b'_j;\beta'_j)_{1,q^{(1)}}, (b^{(r)}_j;\beta^{(r)}_j)_{1,q^{(r)}} \end{matrix} \right)$$

$$= \frac{1}{(2\pi\, w)^r} \int_{L_1} .... \int_{L_r} \xi(s_1, ..., s_r) \prod_{i=1}^{r} \phi(s_i)\, {z_i}^{s_i} ds_1 ....ds_r$$

where $\xi(s_1, ..., s_r)$ and $\phi(s_i)$ clearly mentioned in [6]. The MMIF is analytic if

(1.2)
$$\sum_{k=1}^{p_2} \alpha^{(i)}_{2k} + \sum_{k=1}^{p_3} \alpha^{(i)}_{3k} + ... + \sum_{k=1}^{p_s} \alpha^{(i)}_{sk} - \sum_{k=1}^{q_2} \beta^{(i)}_{2k} - \sum_{k=1}^{q_3} \beta^{(i)}_{3k} - ... - \sum_{k=1}^{q_s} \beta^{(i)}_{sk} - \sum_{j=1}^{R} f^{(i)}_j \le 0$$

The contour integral in (1.1) converges absolutely if $|\arg z_i| < \frac{1}{2}\Omega_i\pi$, where

(1.3) $$\Omega_i = \sum_{k=1}^{n^{(i)}} \alpha^{(i)}_k - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha^{(i)}_k + \sum_{k=1}^{m^{(i)}} \beta^{(i)}_k - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta^{(i)}_k + \sum_{k=1}^{n_2} \alpha^{(i)}_{2k} - \sum_{k=n_2+1}^{p_2} \alpha^{(i)}_{2k}$$

$$+ \sum_{k=1}^{n_3} \alpha^{(i)}_{3k} - \sum_{k=n_3+1}^{p_3} \alpha^{(i)}_{3k} + ... + \sum_{k=1}^{n_r} \alpha^{(i)}_{rk} - \sum_{k=n_r+1}^{p_r} \alpha^{(i)}_{rk}$$

$$- \sum_{k=1}^{q_2} \beta^{(i)}_{2k} - \sum_{k=1}^{q_3} \beta^{(i)}_{3k} .... - \sum_{k=1}^{q_r} \beta^{(i)}_{rk} + \sum_{j=1}^{R'} g^{(i)}_j - \sum_{j=1}^{R} f^{(i)}_j > 0 \text{ (i=1,...,r).}$$

We note

(1.4) $$\mathrm{A}{=}(a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};....;\left(a_{(r-1)j};\alpha'_{(r-1)j},....,\alpha^{r-1}_{(r-1)j}\right)_{1,p_{r-1}}$$

(1.5) $$\mathrm{B}=\left(b_{2j};\beta'_{2j},\beta''_{2j}\right)_{1,q_2};....;\left(b_{(r-1)j};\beta'_{(r-1)j},....,\beta^{r-1}_{(r-1)j}\right)_{1,q_{r-1}}$$

(1.6) $$\mathbf{A}=\left(a_{rj};\alpha'_{rj},....,\alpha^{(r)}_{rj}\right)_{1,p_r};\Im=(a'_j,\alpha'_j)_{1,p'};....;\left(a^{(r)}_j,\alpha^{(r)}_j\right)_{1,p^{(r)}}$$

(1.7) $$\mathbf{B}=\left(b_{rj};\beta'_{rj},....,\beta^{(r)}_{rj}\right)_{1,q_r};\Re=\left(b'_j,\beta'_j\right)_{1,q'};....;\left(b^{(r)}_j,\beta^{(r)}_j\right)_{1,q^{(r)}}$$

$$\mathbf{E}=\left(e_j;u'_jg'_j,....,u^{(r)}_jg^{(r)}_j\right)_{1,R'};L=\left(l_j;U'_jf'_j,....,U^{(r)}_jf^{(r)}_j\right)_{1,R}$$

(1.8) $$U=p_2,q_2;p_3,q_3;...;p_{r-1},q_{r-1}\,;\ V=0,n_2;0,n_3;...;0,n_{r-1}$$

(1.9) $$Y=(p',q')\,;....\left(p^{(r)},q^{(r)}\right);X=(m',n')\,:....;\left(m^{(r)},n^{(r)}\right)$$

2. Main results

In this section, we establish the summation formulae for the MMIF as follows:

Theorem 2.1.

(2.1)

$$\sum_{u_1,...u_m=0}^{\infty}\prod_{J=1}^{m}\frac{((w_j))_{u_j}}{u_j!}\;I^{V;0,n_r+1:|R':X}_{U;p_r+1,q_r+1:|R:Y}\left(\begin{array}{c|c} z_1 & A; \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B;\mathrm{B}, \end{array}\right.$$
$$\left.\begin{array}{c}(1-g-\sum_{j=1}^{m}t_j;a_1,...,a_r),\mathrm{A}:E:\Im \\ \cdot \\ \cdot \\ (1-h-\sum_{j=1}^{m}t_j;b_1,...,b_r):L:\Re\end{array}\right)$$

$$=I^{V;0,n_r+2:|R':X}_{U;p_r+2,q_r+2:|R:Y}\left(\begin{array}{c|c} z_1 & A;(1-g;a_1,...a_r), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B;\mathrm{B}, \end{array}\right.$$

$$\left.\begin{array}{c}(1+g-h+\sum_{j=1}^{m} w_j; b_1-a_1,...,b_r-a_r), \mathrm{A}: E: \Im \\ . \\ . \\ (1-h+\sum_{j=1}^{m} w_j; b_1,...,b_r), (1+g-h; b_1-a_1,...,b_r-a_r): L: \Re\end{array}\right)$$

Following the lines of Braaksma([4], p.278), we may establish the asymptotic expansion in the following convenient way :

$$a_i, b_i, b_i-a_i>0 (i=1,...,r),\ \mathrm{Re}(h-g-\sum_{j=1}^{m} w_j)>0 \textit{ and } |arg(z_i)|<\tfrac{1}{2}(\Omega_i-2b_i)\pi$$

Proof. To establish the Theorem (2.1), expressing the MMIF by Prasad [15] in the Mellin-Barnes multiple integrals contour using (1.1) and interchanging the order of summation and integration, we obtain

$$I=\frac{1}{(2\pi w)^r}\int_{L_1}...\int_{L_r}\phi(s_1,...,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\frac{\Gamma(g+\sum_{j=1}^{m} a_i s_i)}{\Gamma(h+\sum_{j=1}^{m} b_i s_i)}$$

$$\times\sum_{u_1,...u_m=0}^{\infty}\prod_{J=1}^{m}\frac{((w_j))_{u_j}}{u_j!}\frac{\left(g+\sum_{j=1}^{m} a_i s_i\right)_{\sum_{j=1}^{m} t_j}}{\left(h+\sum_{j=1}^{m} b_i s_i\right)_{\sum_{j=1}^{m} t_j}}ds_1...ds_2$$

Now applying result of Panda([12], p.108, Eq.2) and Gauss's theorem ([10], p.28, Eq.1.7.6) in the above equation and interpreting the resulting expression with the help of (1.1), we arrive at Theorem (2.1).

□

Theorem 2.2.

(2.2)

$$\sum_{u_1,...u_m=0}^{\infty}\prod_{j=1}^{m}\frac{((w_j))_{u_j}}{u_j!}I_{U;p_r+2,q_r+2:|R:Y}^{V;0,n_r+2:|R':X}\left(\begin{array}{c|c}z_1 & A;(1-g-\sum_{j=1}^{m} t_j; a_1,...,a_r), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B;\mathrm{B},\end{array}\right.$$

$$\left.\begin{array}{c}(1-g'-\sum_{j=1}^{m} t_j; a'_1,...,a'_r), \mathrm{A}: E: \Im \\ . \\ . \\ (g'-g-\sum_{j=1}^{m} t_j; a_1-a'_1,...,a_r-a'_r),(\sum_{j=1}^{m} w_j-g-\sum_{j=1}^{m} t_j; a_1,...,a_r): L: \Re\end{array}\right)$$

$$= I^{V;0,n_r+3:|R':X}_{U;p_r+3,q_r+3:|R:Y}\left(\begin{array}{c|c} z_1 & A;(1-\frac{g}{2};\frac{a_1}{2},...,\frac{a_r}{2}),(1-g';a'_1,...,a'_r), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B;\mathrm{B},(g'-\frac{g}{2};\frac{a_1}{2}-a'_1,...,\frac{a_r}{2}-a'_r), \end{array}\right.$$

$$\left.\begin{array}{c} (g'+\sum_{j=1}^m w_j-\frac{a}{2};\frac{a_1}{2}-a'_1,...,\frac{a_r}{2}-a'_r),\mathrm{A}:E:\Im \\ \cdot \\ \cdot \\ (\sum_{j=1}^m w_j+g'-g;a_1-a'_1,...,a_r-a'_r):L:\Re \end{array}\right)$$

provided

$a_i, a'_i, a_i-2a_i>0(i=1,...,r)$, $\mathrm{Re}(g'-\frac{g}{2}-\sum\limits_{j=1}^{m} w_j)>0$ *and* $|arg(z_i)|<\frac{1}{2}(\Omega_i-2a_i)\pi$

Theorem 2.3.

(2.3)

$$\sum_{u_1,...u_m=0}^{\infty}\prod_{j=1}^{m}\frac{((w_j))_{u_j}}{u_j!}\ I^{V;0,n_r+4:|R':X}_{U;p_r+4,q_r+4:|R:Y}\left(\begin{array}{c|c} z_1 & A;(1-g-\sum_{j=1}^m t_j;a_1,...,a_r), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B;\mathrm{B}, \end{array}\right.$$

$$\begin{array}{c} (1-g'-\sum_{j=1}^m t_j;a'_1,...,a'_r),(1-g''-\sum_{j=1}^m t_j;a''_1,...,a''_r), \\ \cdot \\ \cdot \\ (g'-g-\sum_{j=1}^m t_j;a_1-a'_1,...,a_r-a'_r),(\sum_{j=1}^m w_j-g-\sum_{j=1}^m t_j;a_1,...,a_r) \end{array}$$

$$\left.\begin{array}{c} (-\frac{g}{2}-\sum_{j=1}^m t_j;\frac{a_1}{2},...,\frac{a_r}{2}),\mathrm{A}:E:\Im \\ \cdot \\ \cdot \\ (1-\frac{g}{2}-\sum_{j=1}^m t_j;\frac{a_1}{2},...,\frac{a_r}{2}),(g''-g-\sum_{j=1}^m t_j;a_1-a''_1,...,a_r-a''_r):L:\Re \end{array}\right)$$

$$= I^{V;0,n_r+3:|R':X}_{U;p_r+3,q_r+3:|R:Y}\left(\begin{array}{c|c} z_1 & A;(1-g';a'_1,...,a'_r) \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B;\mathrm{B},(g'-g+\sum_{j=1}^m w_j;a_1-a'_1,...,a_r-a'_r), \end{array}\right.$$

$$\begin{array}{c} (1-g'';a''_1,...,a''_r),(g'+g''-g+\sum_{j=1}^m w_j;a_1-a''_1,...,a_r-a''_r), \\ \cdot \\ \cdot \\ (g''-g+\sum_{j=1}^m w_j;a_1-a''_1,...,a_r-a''_r), \end{array}$$

$$\left.\begin{array}{c} \mathrm{A}:E:\Im \\ \cdot \\ \cdot \\ (g'+g''-g+\sum_{j=1}^{m} w_j; a_1-a_1'-a_1'', ..., a_r-a_r'-a_r''):L:\Re \end{array}\right)$$

provided $a_i, a_i', a_i'', a_i-a_i'-a_i''>0 (i=1,...,r)$, $\mathrm{Re}(g'+g''-g-\sum\limits_{j=1}^{m} w_j)<1$ *and* $|arg(z_i)|<\frac{1}{2}(\Omega_i-\frac{7}{2}a_i)\pi$.

Theorem 2.4.

(2.4)

$$\sum_{u_1,...u_m=0}^{\infty}\prod_{j=1}^{m}\frac{((w_j))_{u_j}}{u_j!}\, I^{V;0,n_r+3:|R':X}_{U;p_r+3,q_r+3:|R:Y}\left(\begin{array}{c|c} z_1 & A; (-\frac{g}{2}-\sum_{j=1}^{m} t_j; \frac{a_1}{2}, ..., \frac{a_r}{2}), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B; \mathrm{B}, (1-\frac{g}{2}-\sum_{j=1}^{m} t_j; \frac{a_1}{2}, ..., \frac{a_r}{2}), \end{array}\right.$$

$$\begin{array}{c} (-\frac{g}{2}-\sum_{j=1}^{m} t_j; \frac{a_1}{2}, ..., \frac{a_r}{2}), (1-g-\sum_{j=1}^{m} t_j; a_1, ..., a_r), \\ \cdot \\ \cdot \\ (1-\frac{g}{2}-\sum_{j=1}^{m} t_j; \frac{a_1}{2}, ..., \frac{a_r}{2}), (-g-\sum_{j=1}^{m} t_j+\sum_{j=1}^{m} w_j; a_1, ..., a_r), \end{array}$$

$$\left.\begin{array}{c} (g''-\sum_{j=1}^{m} t_j; a_1'', ..., a_r''), \mathrm{A}:E:\Im \\ \cdot \\ \cdot \\ (g'-g-\sum_{j=1}^{m} t_j; a_1-a_1'', ..., a_r-a_r''):L:\Re \end{array}\right)$$

$$=I^{V;0,n_r+3:|R':X}_{U;p_r+3,q_r+3:|R:Y}\left(\begin{array}{c|c} z_1 & A; \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B; \mathrm{B}, \end{array}\right.$$

$$\begin{array}{c} (\frac{1-g}{2}; \frac{a_1}{2}, ..., \frac{a_r}{2}), (1-g''; a_1'', ..., a_r''), \\ \cdot \\ \cdot \\ (\frac{1-g}{2}+g'; \frac{a_1}{2}-a_1', ..., \frac{a_r}{2}-a_r'), (\frac{1-g}{2}+\sum_{j=1}^{m} w_j; \frac{a_1}{2}, ..., a_r), \end{array}$$

$$\left.\begin{array}{c} (\frac{1-g}{2}+\sum_{j=1}^{m} w_j; \frac{a_1}{2}, ..., \frac{a_r}{2}, \mathrm{A}:E:\Im \\ \cdot \\ \cdot \\ (g'-g+\sum_{j=1}^{m} w_j; a_1-a_1', ..., a_r-a_r'):L:\Re \end{array}\right)$$

provided
$a_i, a_i', a_i - a_i' - 2a_i'' > 0 (i = 1, ..., r)$, $\mathrm{Re}(g' - \frac{g}{2} - \sum\limits_{j=1}^{m} w_j) < \frac{1}{2}$ *and* $|arg(z_i)| < \frac{1}{2}(\Omega_i - \frac{5}{2}a_i)\pi$

To prove Theorems (2.2, 2.3 and 2.4), we follow the similar lines with the help of ([10], p.52, Eq.(2.3.3.5)), ([10], p.56, Eq.(2.3.4.5)) and ([10], p.245, Eq.(III.22)) respectively, instead of Gauss's theorem.

3. Particular Cases

In this section, we observe several particular cases. If we take $a_i' = 0 (i = 1, ..., r)$and assume $g' \to \infty$ in Theorem (2.2) and Theorem (2.4), also using the following properties of confluence,

$$\lim_{\lambda \to \infty} \left[(\lambda)_m \left(\frac{z}{\lambda} \right)^m \right] = z^m \tag{3.1}$$

and

$$\lim_{\rho \to \infty} \left[\frac{(\rho w)^m}{(\rho)_m} \right] = w^m, \; m = 0, 1, ... \tag{3.2}$$

After algebraic simplification, we obtain the following corollaries :

Corollary 3.1.

(3.3)

$$\sum_{u_1, ... u_m = 0}^{\infty} \prod_{j=1}^{m} \frac{(-1)^{t_j} ((w_j))_{u_j}}{u_j!} \; I_{U; p_r+1, q_r+1: |R: Y}^{V; 0, n_r+1: |R': X} \left(\begin{array}{c|c} z_1 & A; \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B; \mathrm{B}, \end{array} \right.$$
$$\left. \begin{array}{c} (1 - g - \sum_{j=1}^{m} t_j; a_1, ..., a_r), \mathrm{A} : E : \Im \\ \cdot \\ \cdot \\ (\sum_{j=1}^{m} w_j - g - \sum_{j=1}^{m} t_j; a_1, ..., a_r) : L : \Re \end{array} \right)$$
$$= I_{U; p_r+1, q_r+1: |R: Y}^{V; 0, n_r+1: |R': X} \left(\begin{array}{c|c} z_1 & A; (1 - \frac{g}{2}; \frac{a_1}{2}, ..., \frac{a_r}{2}), \mathrm{A} : E : \Im \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B; \mathrm{B}, (\sum_{j=1}^{m} w_j - \frac{g}{2}; \frac{a_1}{2}, ..., \frac{a_r}{2}) : L : \Re \end{array} \right)$$

provided $a_i > 0 (i = 1, ..., r)$, $\mathrm{Re}(\sum\limits_{j=1}^{m} w_j) > 0$ *and* $|arg(z_i)| < \frac{1}{2}(\Omega_i - a_i)\pi$.

Corollary 3.2.

(3.4)

$$\sum_{u_1,\ldots u_m=0}^{\infty}\prod_{j=1}^{m}\frac{(-1)^{t_j}(w_j)_{u_j}}{u_j!}\; I^{V;0,n_r+2:|R':X}_{U;p_r+2,q_r+2:|R:Y}\left(\begin{array}{c|c} z_1 & A; \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B;\mathrm{B}, \end{array}\right.$$

$$\left.\begin{array}{c} (-\frac{g}{2}-\sum_{j=1}^{m}t_j;\frac{a_1}{2},...,\frac{a_r}{2}),(1-g-\sum_{j=1}^{m}t_j;a_1,...,a_r),\mathrm{A}:E:\Im \\ \cdot \\ \cdot \\ (1-\frac{g}{2}-\sum_{j=1}^{m}t_j;\frac{a_1}{2},...,\frac{a_r}{2}),(-g-\sum_{j=1}^{m}t_j+\sum_{j=1}^{m}w_j;a_1,...,a_r):L:\Re \end{array}\right)$$

$$=I^{V;0,n_r+1:|R':X}_{U;p_r+1,q_r+1:|R:Y}\left(\begin{array}{c|c} z_1 & A;(\frac{1-g}{2};\frac{a_1}{2},...,\frac{a_r}{2}),\mathrm{A}:E:\Im \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B;\mathrm{B},(\frac{1-g}{2}+\sum_{j=1}^{m}w_j;\frac{a_1}{2},...,\frac{a_r}{2}):L:\Re \end{array}\right)$$

provided $a_i>0(i=1,...,r)$, $\mathrm{Re}(\sum_{j=1}^{m}w_j)<\frac{1}{2}$ *and* $|arg(z_i)|<\frac{1}{2}(\Omega_i-\frac{3}{2}a_i)\pi$.

Taking $a_i=0(i=1,...,r)$ and assume $g''\to\infty$ in Theorem (2.3). Also using the equations (2.4),(3.1) and after algebraic manipulations, we obtain the following corollary.

Corollary 3.3.

(3.5)

$$\sum_{u_1,\ldots u_m=0}^{\infty}\prod_{j=1}^{m}\frac{(-1)^{t_j}(w_j)_{u_j}}{u_j!}\; I^{V;0,n_r+3:|R':X}_{U;p_r+3,q_r+3:|R:Y}\left(\begin{array}{c|c} z_1 & A; \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B;\mathrm{B}, \end{array}\right.$$

$$\begin{array}{c} (1-g-\sum_{j=1}^{m}t_j;a_1,...,a_r),(1-g'-\sum_{j=1}^{m}t_j;a'_1,...,a'_r), \\ \cdot \\ \cdot \\ (g'-g-\sum_{j=1}^{m}t_j;a_1-a'_1,...,a_r-a'_r),(\sum_{j=1}^{m}w_j-g-\sum_{j=1}^{m}t_j;a_1,...,a_r) \end{array}$$

$$\left.\begin{array}{c} (-\frac{g}{2}-\sum_{j=1}^{m}t_j;\frac{a_1}{2},...,\frac{a_r}{2}),\mathrm{A}:E:\Im \\ \cdot \\ \cdot \\ (1-\frac{g}{2}-\sum_{j=1}^{m}t_j;\frac{a_1}{2},...,\frac{a_r}{2}):L:\Re \end{array}\right)$$

$$= I^{V;0,n_r+1:|R':X}_{U;p_r+1,q_r+1:|R:Y}\left(\begin{array}{c|c} z_1 & A;(1-g';a'_1,...,a'_r),\mathrm{A}:E:\Im \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B;\mathrm{B},(g'-g+\sum_{j=1}^{m} w_j;a_1-a'_1,...,a_r-a'_r):L:\Re \end{array}\right)$$

provided $a_i > 0 (i = 1, ..., r)$, $\mathrm{Re}(\sum\limits_{j=1}^{m} w_j) < \frac{1}{2}$
and $|arg(z_i)| < \frac{1}{2}(\Omega_i - \frac{3}{2}a_i)\pi$.

We can give a number of corollaries by specializing the parameters. The multiple summation formulae involved in this article are general in nature in their manifold.

4. Concluding Remarks

If I-function defined by Prasad [15] reduces in generalized form of H-function defined by Prasad and Singh [14], we obtain the similar relations using analogue techniques. Also by modifying the functions defined by Srivastava and Panda([8], [9]) and Goyal and Garg [13], we can obtain similar type of relations.

The importance of all these results are common in nature. We can obtain single, double or multiple summation formulae by making use of general multiple summation formulae used here. By specializing various parameters and variables in the MMIF, we get several useful product of such functions like E, F, G, H and I of one and several variables. These formulae are useful in many interesting cases of Applied Mathematics and Mathematical Physics. In the next extension of this work, we are going to apply these summation formulae to obtain the solutions of Boundary value problems.

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Analysis of unsteady MHD Williamson nanofluid flow past a stretching sheet with nonlinear mixed convection, thermal radiation and velocity slips

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Abstract

This article examines the transient MHD convective flow with heat and mass transport of Williamson nanofluid over a stretching sheet in the presence of a chemical reaction. Velocity slips, convective heating and vanishing mass flux conditions at the surface are imposed. As a novelty, the effects of nonlinear thermal radiation, mixed convection, velocity slips and activation energy are incorporated. Such problems find significant applications in aircraft, missiles, gas turbines, etc. Similarity transformations are employed to convert controlling PDEs into a system of ODEs and the resulting nonlinear BVP is solved numerically using *bvp4c*. The effects of various parameters on velocity, temperature and concentration distributions are demonstrated and depicted graphically. However, the numerical values of local skin friction coefficients, Nusselt and Sherwood numbers are tabulated and discussed. The graphs show that the nonlinear convection parameters, for both temperature and concentration, boost the primary flow. Higher values of the velocity slip parameters result in diminishing the flow. The fluid temperature rises as a result of both radiation and convective heating. The activation energy improves the concentration profile within the boundary layer. The current findings would appeal to a broad audience in mechanical engineering, medical sciences, industrial engineering, etc.

Keywords Williamson nanofluid · Thermal radiation · Velocity slip · Convective heating · Activation energy · Chemical reaction

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Nomenclature

u, v, w	Velocity components	L	Dimensionless quantity
x, y, z	Space coordinate	N	Buoyancy ratio parameter
T_∞	Ambient temperature	Le	Lewis number
C_∞	Ambient concentration	D_T	Thermal diffusion coefficient
S	Unsteadiness parameter	Bi	Biot number
n	Fitted rate constant		
kr	Chemical reaction coefficient	**Greek Symbols**	
$u_{\mathrm{w}}, v_{\mathrm{w}}$	Stretching velocities along x and y directions	ν	Kinetic viscosity
		μ	Dynamic viscosity
B_0	Constant magnetic field	Γ_1	Chemical reaction parameter
k	Thermal conductivity	σ	Electrical conductivity
D_B	Mass diffusivity	κ	Boltzmann constant
Rd	Thermal radiation parameter	σ^*	Stefan-Boltzmann constant
Pr	Prandtl number	θ	Dimensionless temperature
g	Gravitational acceleration	Γ	Material parameter
c_p	Specific heat	α	Thermal diffusivity
T	Fluid Temperature	ϕ	Dimensionless concentration
C	Species concentration	α_1, α_2	Velocity slip parameters
Nt	Thermophoresis parameter	λ_1, λ_2	Nonlinear thermal and solutal convection parameters
T_{w}	Wall temperature		
C_{w}	Wall concentration	ρ	Fluid density
Nb	Brownian motion parameter	θ_{w}	Temperature ratio parameter
d_1^*, d_2^*	Velocity slip coefficients	β_1	Velocity ratio parameter
We_1, We_2	Weissenberg numbers along x and y directions	τ	Heat capacity ratio
		β_C, β_C^*	Linear and non-linear solutal expansion coefficients
E_a	Activation energy		
M	Magnetic parameter	β_T, β_T^*	Linear and non-linear thermal expansion coefficients
k^*	Mean absorption coefficient		
E	Dimensionless activation energy	λ	Mixed convection parameter

1 Introduction

For the past few decades, nanotechnology-based techniques have been used to create nanoscale particles with a size of less than 100 nm. Stable suspensions can be made using nanoparticles to increase the thermal characteristics of the base fluid. It has been demonstrated that adding tiny quantities of metal or metal oxide nanoparticles to liquid improves thermal conductivity. Nanofluids, like current working fluids, have high heat absorption and heat transmission characteristics. Recent years have seen a significant increase in interest in nanofluids research owing to its numerous usages in communication, electronics and computer systems, as well as optical devices. Hayat et al. [1] discussed the movement of a non-Newtonian fluid across a wedge as a mixed convection flow.

Nourazar et al. [2] used the HPM to solve an MHD nanofluid flow on a horizontal flat plate with a changing magnetic field and viscous dissipation. A study on the effect of natural convection in viscoelastic fluid past a cone taking viscous dissipation was done by Makanda et al. [3]. In a rotating device, Sheikholeslami et al. [4] examined the nanofluid flow and heat transmission properties between two parallel horizontal plates. Using a fixed wedge with changing wall temperature and concentration, Srinivasacharya et al. [5] investigated the influence of a varied magnetic field on nanofluid flow.

Understanding the boundary layer flow with heat transfer along a stretched sheet has become more significant because of several engineering activities. Extrusion of polymers, paper manufacturing, and other similar processes are examples of chemical engineering and metallurgy applications. The rate of heat transfer between the fluid and stretching surface considering heating and/or cooling has a significant impact on the quality of the final product. As a result, the choice of heating or cooling fluid is critical to the heat transfer rate. In light of the physical relevance of heat transmission across moving surfaces, several researchers have been obliged to publish their discoveries in this area. Crane [6] examined the flow past a stretched plate that is subject to the relation between the velocity and the distance from a slit. This yielded an accurate result. Following Crane's work, MHD viscous flow across a stretched sheet was given by Azimi et al. [7], who discussed the analysis of momentum features in the flow. Dessie and Kishan [8] investigated the effect of viscous dissipation and heat source/sink over a stretching sheet. Mishra et al. [9] studied numerically MHD power-law fluid flow over a stretching sheet taking a non-uniform heat source.

Regarding the MHD heat transfer fluxes, thermal radiation is a crucial factor in controlling heat transfer rate. It may impact many industrial processes such as glass manufacture, gas turbine production, furnace design, and re-entry vehicle engine design. As a result, this generated extensive studies on the influence of heat radiation in hydromagnetic fluxes. Daniel and Daniel [10] explored the impact of thermal radiation and buoyancy force on MHD flow through a stretchable sheet with the help of the homotopy analysis method. Kumbhakar and Rao [11] discussed MHD stagnation point flow of an electrically conducting fluid over a nonlinearly stretching surface considering thermal radiation and viscous dissipation. Kho et al. [12] studied thermal radiation effect in the flow of Williamson nanofluid passing through a stretching sheet. With heat and mass transfer through an unstable stretched surface in a uniform magnetic field, Ishaq et al. [13] explored entropy production and thermal radiation. Alharbi et al. [14] conducted experiments on MHD Eyring-Powell flow in an unstable oscillatory stretching sheet to evaluate the influence of thermal radiation and a heat source/sink. Kumar et al. [15] examined the transient natural/free convective nanofluid flow past a vertical plate with effects of radiation and magnetic field.

According to current trends in chemical reaction analysis, it is essential to create a mathematical model of a system to forecast its performance. Especially in the chemical and hydro-metallurgical sectors, heat and mass transport research during chemical reactions is of great significance. Some examples of

combined heat and mass transfer applications with chemical reaction effects are chemical processing equipment design, fog formation and dispersion, temperature and moisture distribution over agricultural fields and fruit tree groves, crop damage due to freezing, cooling towers, and food processing. An excellent example of a first-order homogeneous chemical reaction is the production of smog. Das [16] examined the effects of thermal radiation and chemical reaction on MHD micropolar fluid flow near an inclined porous plate. Sheikh and Abbas [17] studied chemical reaction impact on MHD viscous fluid flow over an oscillating stretching sheet under the influence of heat generation/absorption. Tarakaramu and Narayan [18] explored the effect of chemical reactions on unsteady MHD nanofluid flow towards a stretchable sheet. Kumar et al. [19] investigated the influence of binary chemical reaction with Arrhenius activation energy on the MHD Carreau fluid flow over a stretched surface. They found that the chemical reaction has a significant impact on the flow. Khan et al. [20] studied the aspects of activation energy and thermal radiation on MHD flow containing Ti_6Al_4V nanoparticle past a stretching sheet. Chu et al. [21] discussed the action of a chemical reaction and activation energy on MHD third grade nanofluid flow past a stretching sheet.

The assumption that the flow field obeys the standard no-slip condition at the sheet is quite common in the preceding research and all relevant references. However, the no-slip criterion is inadequate when the fluid is made up of particle emulsions and polymers. Furthermore, boundary-slipping fluids have crucial technological uses, such as cleaning prosthetic heart valves and interior cavities. In such circumstances, the partial slip is an appropriate boundary condition. Additionally, when micro-scale dimensions are included in the flow field, such a slip is necessary. Slip at the wall boundary significantly alters the fluid's flow behavior and shear stress than no-slip circumstances. Using a low-magnetic Reynolds number assumption, Zheng et al. [22] investigated the slip consequences of Oldroyd-B fluid flow across a plate. Hayat et al. [23] explored velocity slip condition on MHD nanofluid flow past a rotating disk. Amanulla et al. [24] discussed the slip effects on MHD Prandtl flow past an isothermal sphere in a non-Darcy porous medium. Ellahi et al. [25] analyzed the combined impact of slip and entropy generation on MHD flow through a moving plate. Khan et al. [26] explored the significance of slip conditions for a magnetite Jeffrey nanofluid flow over a porous stretching sheet in the existence of thermal radiation and the Soret effect. Das et al. [27] studied mutiple slip effects on tangent hyperbolic fluid flow along a stretching sheet considering Soret and Dufour effects, thermal radiation and heat source.

In processes in which high temperatures are involved, convective heat transfer is essential. Consider the following examples: gas turbines, nuclear power plants, thermal energy storage, and so forth. It is more feasible to use convective boundary conditions in industrial and technical processes, such as material drying and transpiration cooling operations [28]. Because of the practical significance of convective boundary conditions in viscous fluids, Many scholars have investigated and presented their findings on this issue. Ramzan et al. [29] investigated the impact of convective heating conditions and Cattaneo-Christov heat

flux with heat production/absorption on MHD 3D Maxwell fluid flow across a bidirectional stretching surface. Nayak et al. [30] studied MHD nanofluid flow over a linearly stretching sheet considering the convective heating boundary constraint along with viscous dissipation, velocity slip, nonlinear thermal radiation and Joule heating. Shah et al. [31] observed simultaneous effects of convective boundary condition and thermal radiation on MHD Carbon nanotubes nanofluid flow across a stretching sheet. Aspects of convective boundary condition, Joule heating, thermal radiation, and a changing heat source/sink were studied in detail by Kumar et al. [32] concerning the flow and heat transfer properties of an electrically conducting Casson fluid due to an exponentially expanding curved surface. Loganathan et al. [33] examined the impact of convective heating, Cattaneo-Christov double diffusion and thermal radiation on MHD Maxwell fluid flow along an extended surface. Recently, Jamshed and Nisar [34] studied convective heating, thermal radiation and heat source effects on Williamson nanofluid flow over a stretching sheet.

Based on the above literature survey, the authors have found that no attempt has been made yet to study the impacts of nonlinear thermal radiation and Arrhenius activation energy on unsteady mixed convective flow of Williamson nanofluid over a stretching surface. Therefore, this research aims to fill such gap by exploring the novel circumstances of nonlinear thermal radiation and activation energy on unsteady MHD convective flow with heat and mass transport of Williamson nanofluid over a stretching sheet in the presence of a chemical reaction. The outcomes of this study may have significant bearings on several practical applications such as in aircraft, missiles, gas turbines, food processing, etc. Numerical solutions are obtained for the velocity, temperature and concentration distributions with the help of *bvp4c* routine of MATLAB software. The impacts of significant flow parameters on velocity, temperature and concentration profiles are illustrated and presented graphically. However, the variations in surface drag-coefficients, Nusselt and Sherwood numbers are discussed using numerical data. Moreover, for a limiting case of the present study, a data comparison is made just to ensure that the obtained results are correct and reliable.

2 Mathematical formulation

Consider a three-dimensional, unsteady and incompressible MHD Williamson nanofluid flow along a stretching surface with velocity slip. Further, the influences of nonlinear thermal radiation and chemical reaction with activation energy are also considered. A physical configuration of the flow problem is demonstrated in Fig. 1. The figure shows that the sheet is positioned in the Cartesian coordinate system (x, y, z) such that the x-axis is along the surface in the direction of flow, y-axis is along the width of the surface, and z-axis is normal to xy plane. A constant magnetic field of magnitude B_0 is applied along the z-direction. The surface is stretched along x and y-directions with velocities $u_{\mathrm{w}} = \frac{ax}{1-\beta t}$ and $v_{\mathrm{w}} = \frac{by}{1-\beta t}$ (a, b being positive constants and β is a parameter

having dimension time^{-1}) respectively. The nanofluid temperature and species concentration at the surface are kept at constant values of T_{w} and C_{w} respectively. In contrast, the ambient fluid temperature and species concentration are maintained at constant values of T_∞ and C_∞, respectively.

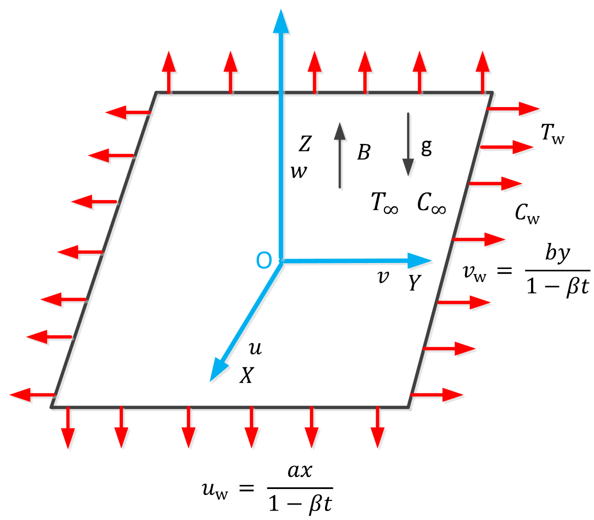

Figure 1: Physical configuration of the problem

Based on the aforementioned assumptions, the governing equations of the current fluid flow (continuity, momentum, energy and species concentration) may be modeled as ([35], [36]):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1}$$

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= \nu\frac{\partial^2 u}{\partial z^2} + \sqrt{2}\nu\Gamma\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} + \frac{\nu\Gamma}{\sqrt{2}}\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2} - \frac{\sigma B^2(t)}{\rho_f}u \\ &+ \mathrm{g}\left[\beta_T(T-T_\infty) + \beta_T^*(T-T_\infty)^2 + \beta_C(C-C_\infty) + \beta_C^*(C-C_\infty)^2\right],\end{aligned} \tag{2}$$

$$\begin{aligned}\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} =&\nu\frac{\partial^2 v}{\partial z^2} + \sqrt{2}\nu\Gamma\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2} + \frac{\nu\Gamma}{\sqrt{2}}\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} \\ &- \frac{\sigma B^2(t)}{\rho_f}v,\end{aligned} \tag{3}$$

6

$$\begin{aligned}\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=&\alpha\frac{\partial^2 T}{\partial z^2}+\tau\left\{D_B\frac{\partial T}{\partial z}\frac{\partial C}{\partial z}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial z}\right)^2\right\}\\&-\frac{1}{(\rho c_p)_f}\frac{\partial q_r}{\partial z},\end{aligned}\tag{4}$$

$$\begin{aligned}\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=&D_B\frac{\partial^2 C}{\partial z^2}+\frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial z^2}\\&-kr\left(C-C_\infty\right)\left(\frac{T}{T_\infty}\right)^n e^{-\frac{E_a}{\kappa T}}.\end{aligned}\tag{5}$$

The physical boundary conditions for the current problem are given as follows:

$$\left.\begin{aligned}&u=u_\mathrm{w}+d_1^*\frac{\partial u}{\partial z},\ v=v_\mathrm{w}+d_2^*\frac{\partial v}{\partial z},\ w=0,\ -k\frac{\partial T}{\partial z}=h_f\left(T_\mathrm{w}-T\right),\\&\quad D_B\frac{\partial C}{\partial z}+\frac{D_T}{T_\infty}\frac{\partial T}{\partial z}=0,\ \text{at } z=0,\\&u\to 0,\quad v\to 0,\quad T\to T_\infty,\quad C\to C_\infty\quad \text{as}\quad z\to\infty.\end{aligned}\right\}\tag{6}$$

In order to approximate the radiative heat flux q_r, the following Rosseland's approximation for an optically thick fluid is employed (Fatunmbi and Adeniyan [37]):

$$q_r=-\frac{16\sigma^* T^3}{3k^*}\frac{\partial T}{\partial z}.\tag{7}$$

The energy equation has the form after applying expression (7) to equation (4)

$$\begin{aligned}\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=&\alpha\frac{\partial^2 T}{\partial z^2}+\tau\left\{D_B\frac{\partial T}{\partial z}\frac{\partial C}{\partial z}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial z}\right)^2\right\}\\&+\frac{16\sigma^* T^2}{3\left(\rho c_p\right)_f k^*}\left\{T\frac{\partial^2 T}{\partial Z^2}+3\left(\frac{\partial T}{\partial Z}\right)^2\right\}.\end{aligned}\tag{8}$$

The variable aspects of wall temperature, wall concentration and magnetic field are given by the following form [38]

$$T_\mathrm{w}(x,t)=\frac{T_0 x u_\mathrm{w}}{\nu(1-\beta t)^{\frac{1}{2}}}+T_\infty,\quad C_\mathrm{w}(x,t)=\frac{C_0 x u_\mathrm{w}}{\nu(1-\beta t)^{\frac{1}{2}}}+C_\infty,\quad B(t)=\frac{B_0}{(1-\beta t)^{\frac{1}{2}}}.$$

To obtain similar solutions of equations (2), (3), (8) and (5) subject to the boundary conditions (6), the following similarity variables are introduced:

$$\left.\begin{aligned}&u=\frac{ax}{1-\beta t}f'(\eta),\quad v=\frac{ay}{1-\beta t}g'(\eta),\quad w=-\sqrt{\frac{a\nu}{1-\beta t}}\left\{f(\eta)+g(\eta)\right\},\\&\theta(\eta)=\frac{T-T_\infty}{T_\mathrm{w}-T_\infty},\quad \phi(\eta)=\frac{C-C_\infty}{C_\mathrm{w}-C_\infty},\quad \eta=z\sqrt{\frac{a}{\nu(1-\beta t)}}.\end{aligned}\right\}\tag{9}$$

7

Substitution of the above similarity variables in equations (2), (3), (8) and (5) yields the following ordinary differential equations:

$$\begin{aligned} & f'''\left[1+We_1 f''\right]+\frac{We_1}{2}L^2 g'' g'''-f'^2+(f+g)\,f''-S\left(f'+\frac{1}{2}\eta f''\right) \\ & -Mf'+\lambda\left(1+\lambda_1\theta\right)\theta+\lambda N\left(1+\lambda_2\phi\right)\phi=0, \end{aligned} \tag{10}$$

$$\begin{aligned} & g'''\left[1+We_2 g''\right]+\frac{We_2}{2L^2}f'' f'''-g'^2+(f+g)\,g''-S\left(g'+\frac{1}{2}\eta g''\right) \\ & -Mg'=0, \end{aligned} \tag{11}$$

$$\begin{aligned} & \theta''+Pr\,(f+g)\,\theta'-Pr\frac{S}{2}(3\theta+\eta\theta')-2Pr\theta f'++PrNb\theta'\phi'+PrNt\theta'^2 \\ & +Rd\left\{1+\theta\left(\theta_{\mathrm{w}}-1\right)\right\}^2\left[3\theta'^2\left(\theta_{\mathrm{w}}-1\right)+\left\{1+\theta\left(\theta_{\mathrm{w}}-1\right)\right\}\theta''\right]=0, \end{aligned} \tag{12}$$

$$\begin{aligned} & \phi''+PrLe\,(f+g)\,\phi'-PrLe\frac{S}{2}(3\phi+\eta\phi')-2PrLe\phi f'+\frac{Nt}{Nb}\theta'' \\ & -PrLe\Gamma_1\left\{1+(\theta_{\mathrm{w}}-1)\theta\right\}^n e^{\left(-\frac{E}{1+(\theta_{\mathrm{w}}-1)\theta}\right)}\phi=0. \end{aligned} \tag{13}$$

The dimensionless boundary conditions are stated as

$$\left.\begin{aligned} & f'(0)=1+\alpha_1 f''(0),\quad g'(0)=\beta_1+\alpha_2 g''(0),\quad f(0)=0,\quad g(0)=0, \\ & \qquad \theta'(0)=-Bi\left(1-\theta(0)\right),\quad \phi'(0)=-\frac{Nt}{Nb}\theta'(0), \\ & f'(\infty)\to 0,\quad g'(\infty)\to 0,\quad \theta(\infty)\to 0,\quad \phi(\infty)\to 0. \end{aligned}\right\} \tag{14}$$

where

$$\begin{aligned} & We_1=\sqrt{\frac{2\Gamma^2 a u_{\mathrm{w}}^2}{\nu(1-\beta t)}},\quad We_2=\sqrt{\frac{2\Gamma^2 a v_{\mathrm{w}}^2}{\beta_1^2\nu(1-\beta t)}},\quad N=\frac{\beta_C(C_{\mathrm{w}}-C_\infty)}{\beta_T(T_{\mathrm{w}}-T_\infty)},\quad S=\frac{\beta}{a}, \\ & \lambda=\frac{\beta_T \mathrm{g}(1-\beta t)(T_{\mathrm{w}}-T_\infty)}{a u_{\mathrm{w}}},\quad M=\frac{\sigma B_0^2}{a\rho_f},\quad L=\frac{y}{x},\quad Pr=\frac{\nu}{\alpha},\quad Le=\frac{\alpha}{D_B}, \\ & Nb=\frac{\tau D_B\left(C_{\mathrm{w}}-C_\infty\right)}{\nu},\quad Nt=\frac{\tau D_T\left(T_{\mathrm{w}}-T_\infty\right)}{\nu T_\infty},\quad \theta_{\mathrm{w}}=\frac{T_{\mathrm{w}}}{T_\infty},\quad E=\frac{E_a}{\kappa T_\infty}, \\ & \Gamma_1=\frac{kr(1-\beta t)}{a},\quad \alpha_1=d_1^*\sqrt{\frac{a}{\nu(1-\beta t)}},\quad \alpha_2=d_2^*\sqrt{\frac{a}{\nu(1-\beta t)}},\quad \beta_1=\frac{b}{a}, \\ & Bi=\frac{h_f}{k}\sqrt{\frac{\nu(1-\beta t)}{a}},\quad \lambda_1=\frac{\beta_T^*\left(T_{\mathrm{w}}-T_\infty\right)}{\beta_T},\quad \lambda_2=\frac{\beta_C^*\left(C_{\mathrm{w}}-C_\infty\right)}{\beta_C} \\ & \delta=\frac{Q_1(1-\beta t)}{a(\rho c_p)_f},\quad Rd=\frac{16\sigma^* T_\infty^3}{3(\rho c_p)_f\alpha k^*}. \end{aligned}$$

3 Skin-friction coefficients, Nusselt number and Sherwood number

The physical quantities of engineering interest for the present fluid flow problem are the local skin-friction coefficients, Nusselt number and Sherwood number. The skin-friction coefficient measures the shear stress, whereas the Nusselt number and Sherwood number describe the rate of heat and mass transfer at the surface. A low Nusselt number signifies that conductive heat transport is more than the convective heat transfer, whereas a high Nusselt number indicates that convective heat transfer dominates the conductive heat transfer. Thermal engineering devices may be designed more effectively with this in mind. Convective mass transfer is divided by diffusive mass transport, and this ratio is known as the Sherwood number. It is used to conduct mass transfer analyses on systems such as liquid-liquid extraction. Mathematically, the local skin-friction coefficients (C_{fx}, C_{fy}), Nusselt number (Nu_x) and Sherwood number (Sh_x) are expressed as

$$C_{fx} = \frac{\nu}{u_{\mathrm{w}}^2}\left[\frac{\partial u}{\partial z}\left\{1+\frac{\Gamma}{\sqrt{2}}\sqrt{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2}\right\}\right]_{z=0}, \tag{15}$$

$$C_{fy} = \frac{\nu}{v_{\mathrm{w}}^2}\left[\frac{\partial v}{\partial z}\left\{1+\frac{\Gamma}{\sqrt{2}}\sqrt{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2}\right\}\right]_{z=0}, \tag{16}$$

$$Nu_x = -\frac{x}{k\left(T_{\mathrm{w}}-T_\infty\right)}\left[\left(k+\frac{16\sigma^* T^3}{3k^*}\right)\frac{\partial T}{\partial z}\right]_{z=0}, \tag{17}$$

$$Sh_x = -\frac{xD_B}{D_B\left(C_{\mathrm{w}}-C_\infty\right)}\left(\frac{\partial C}{\partial z}\right)_{z=0}. \tag{18}$$

The aforementioned physical values can be expressed in non-dimensional form using the dimensionless variables specified in (9)

$$C_{fx}\sqrt{Re_x} = f''(0)\left[1+\frac{We_1}{2}\sqrt{f''^2(0)+L^2 g''^2(0)}\right], \tag{19}$$

$$C_{fy}\sqrt{Re_y} = g''(0)\left[1+\frac{We_2}{2}\sqrt{\frac{1}{L^2}f''^2(0)+g''^2(0)}\right]\sqrt{\beta_1^{-3}}, \tag{20}$$

$$\frac{Nu_x}{\sqrt{Re_x}} = -\left[1+Rd\left\{1+\left(\theta_w-1\right)\theta(0)\right\}^3\right]\theta'(0), \tag{21}$$

$$\frac{Sh_x}{\sqrt{Re_x}} = -\phi'(0), \tag{22}$$

where $Re_x = \dfrac{u_{\mathrm{w}}x}{\nu}$ and $Re_y = \dfrac{v_{\mathrm{w}}y}{\nu}$ are the local Reynolds numbers.

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4 Numerical solution

4.1 Methodology

The coupled and highly nonlinear ordinary differential equations (10)-(13) subject to the boundary conditions (14) are solved numerically by employing the *bvp4c* solver in MATLAB. The higher-order equations (10)-(13) are converted into a set of first-order equations. Furthermore, while implementing the numerical technique, the boundary value problem is metamorphosed into an initial value problem by assuming some suitable guess values to those missing initial conditions.

Table 1: Comparison of values of $-f''(0)$ for altered values of M when $\beta_1 = 0.5$

	$-f''(0)$		
M	Present	Oyelakin et al. [39]	Nadeem et al. [40]
0	1.093096	1.09310	1.0932
10	3.342030	3.34204	3.3420
100	10.058166	10.05818	10.058

Table 2: Comparison of values of $-g''(0)$ for altered values of M when $\beta_1 = 0.5$

	$-g''(0)$		
M	Present	Oyelakin et al. [39]	Nadeem et al. [40]
0	0.465206	0.46520	0.4653
10	1.645891	1.64590	1.6459
100	5.020785	5.02080	5.0208

4.2 Validation

The numerical values of $-f''(0)$ and $-g''(0)$ displayed in Tables 1 and 2 have been computed for different values of magnetic parameter M for a specific situation of the current problem, i.e., when $We_1 = We_2 = \lambda = \lambda_1 = \lambda_2 = \alpha_1 = \alpha_2 = N = 0$, $\beta_1 = 0.5$ and $n = 1$ to test the correctness of the obtained results and the reliability of the employed numerical approach. From the tables, it is clearly observed that our results have a firm agreement with the results reported by Oyelakin et al. [39] and Nadeem et al. [40].

5 Results and discussion

This section presents the analysis of the obtained results for the current heat and mass transport phenomenon. The behavior of the flow profiles as well as the physical quantities of practical importance, is investigated in depth with respect to the changes of the emergent parameters. For the computational purpose, we have assumed the parameters' values as $We_1 = We_2 = S = 0.2$,

$N = n = Nt = 0.5$, $Pr = \theta_{\mathrm{w}} = 1.2$, $L = Nb = \lambda = \alpha_1 = \alpha_2 = 0.4$, $Rd = 0.1$, $Le = M = 1.0$, $\beta_1 = 0.7$, $Bi = \lambda_1 = \lambda_2 = K_1 = 0.3$, $E = 0.6$. Throughout the study, the same values for parameters are adopted, while the altered values of the parameters are shown separately in the respective figures.

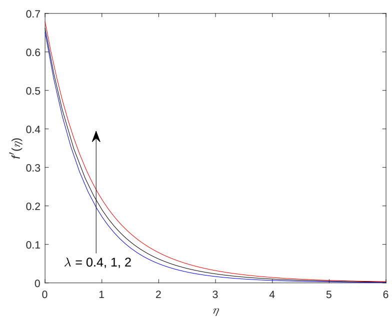

Figure 2: Changes in $f'(\eta)$ vs λ

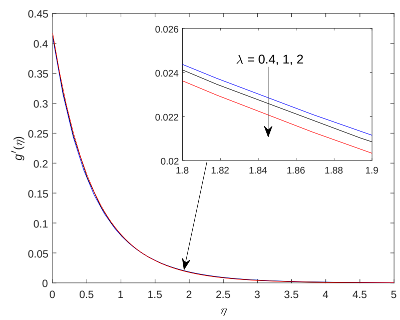

Figure 3: Changes in $g'(\eta)$ vs λ

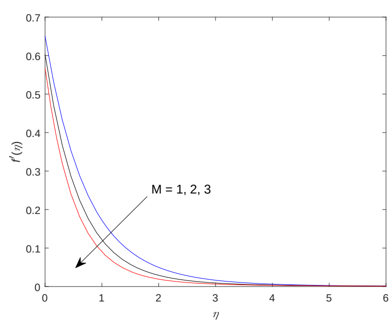

Figure 4: Changes in $f'(\eta)$ vs M

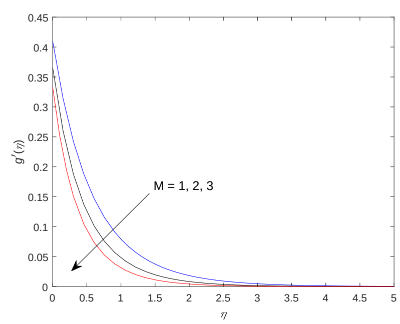

Figure 5: Changes in $g'(\eta)$ vs M

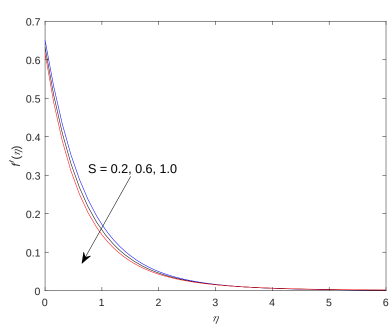

Figure 6: Changes in $f'(\eta)$ vs S

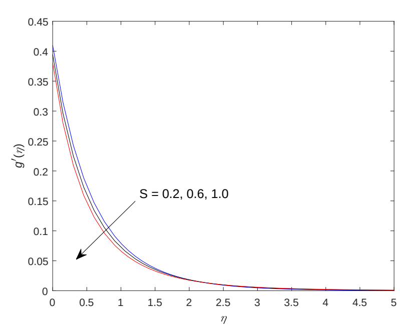

Figure 7: Changes in $g'(\eta)$ vs S

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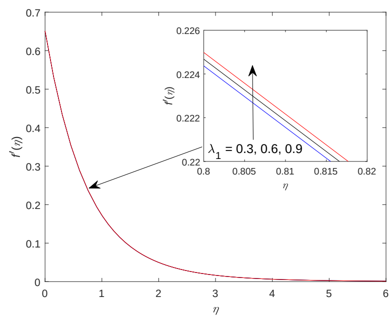

Figure 8: Changes in $f'(\eta)$ vs λ_1

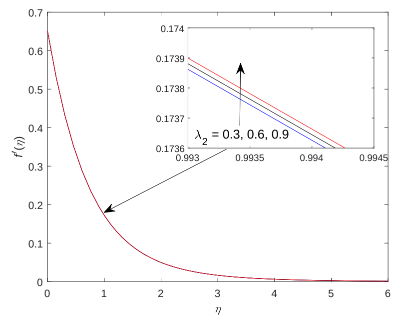

Figure 9: Changes in $f'(\eta)$ vs λ_2

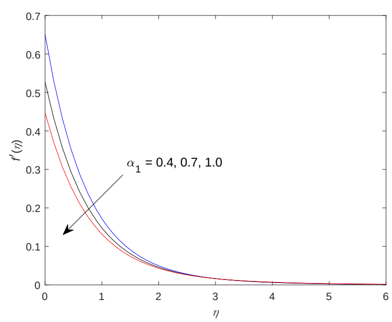

Figure 10: Changes in $f'(\eta)$ vs α_1

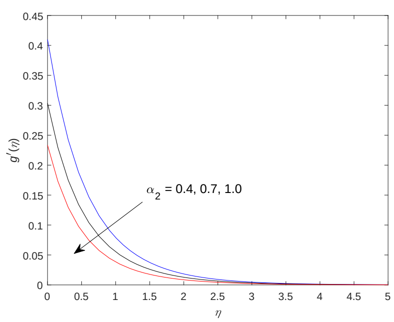

Figure 11: Changes in $g'(\eta)$ vs α_2

Figures 2-11 illustrate the influence of λ, M, S, λ_1, λ_2, α_1 and α_2 on the velocity field. Growth in $f'(\eta)$ and reduction in $g'(\eta)$ are observed in Figures 2 and 3. The higher mixed convection parameter contributes to a larger buoyancy force. This powerful force accelerates the primary flow by suppressing the flow in the secondary direction. A significant increase in the magnetic parameter has caused a significant drop in the nanofluid velocity profile. Increased M leads to a corresponding rise in the resistive Lorentz force, which causes the fluid flow to decrease as depicted in Figures 4 and 5. Decreasing nature of $f'(\eta)$ and $g'(\eta)$ for improvement in S is noted in Figures 6 and 7. In Figures 8 and 9, it is noticed that larger values of λ_1 and λ_2 indicate an upsurge in $f'(\eta)$. Temperature and concentration differences arise from nonlinear convection parameters λ_1 and λ_2 that are greater than the equivalent linear convection values. Velocity is therefore emphasized. Figures 10 and 11 express diminishing character of $f'(\eta)$ and $g'(\eta)$ w.r.t. α_1 and α_2. An increase in velocity slip parameters lead to increase the slip between the fluid and surface of the sheet. So a partial slip velocity moved to the flow field that has the tendency to decelerate the flow.

Figure 12 shows that $\theta(\eta)$ heightens on rising values of M. When the magnetic parameter increases, a stronger Lorentz force is generated. This force

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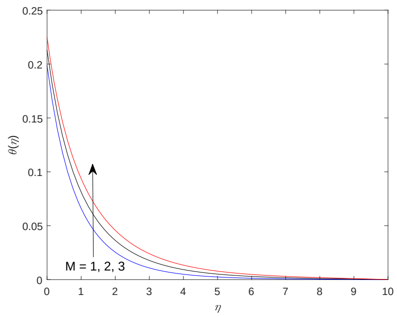

Figure 12: Changes in $\theta(\eta)$ vs M

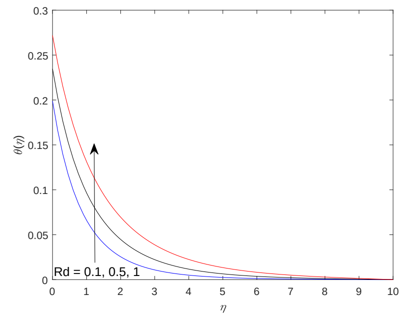

Figure 13: Changes in $\theta(\eta)$ vs Rd

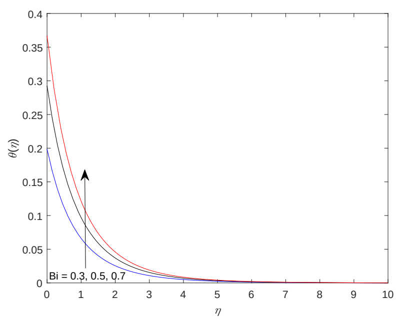

Figure 14: Changes in $\theta(\eta)$ vs Bi

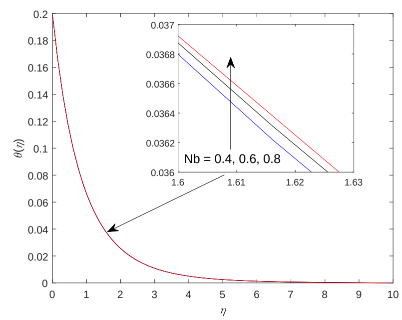

Figure 15: Changes in $\theta(\eta)$ vs Nb

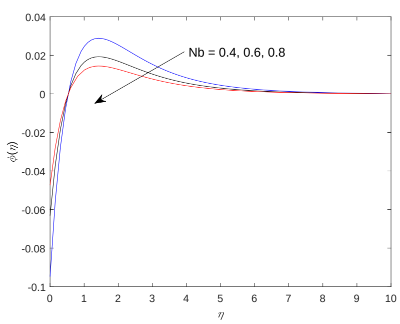

Figure 16: Changes in $\phi(\eta)$ vs Nb

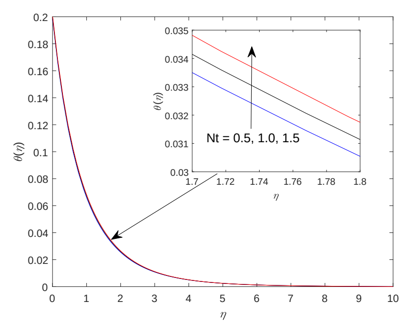

Figure 17: Changes in $\theta(\eta)$ vs Nt

provides resistance against the flow and thereby, the fluid temperature is intensified. Figure 13 elucidates a rising trend for $\theta(\eta)$ on enhanced values of Rd. Improved radiation parameter reduces the mean heat absorption coefficient. As a result, the fluid temperature gets hiked. From Figure 14, an increase in $\theta(\eta)$ is noticed for enlarged values of Bi. An increase in the Biot number leads to

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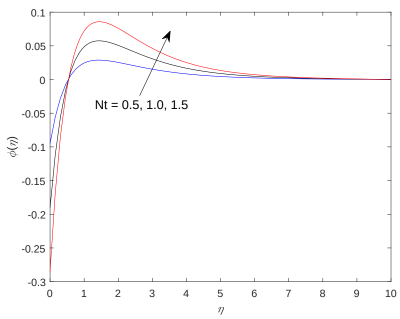

Figure 18: Changes in $\phi(\eta)$ vs Nt

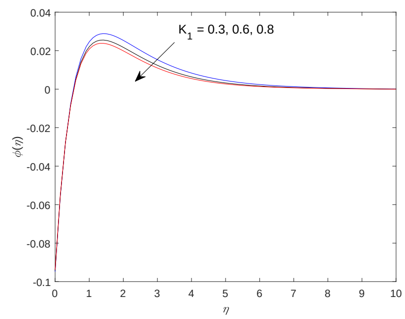

Figure 19: Changes in $\phi(\eta)$ vs K_1

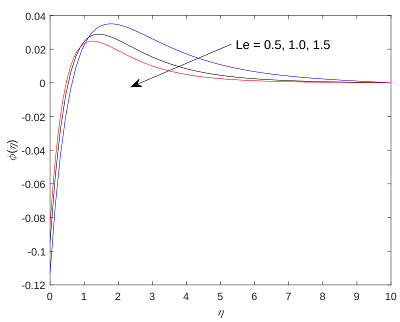

Figure 20: Changes in $\phi(\eta)$ vs Le

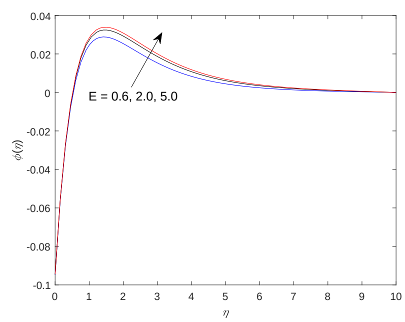

Figure 21: Changes in $\phi(\eta)$ vs E

enhance the heat transfer due to convective heating with hot fluid. So, temperature of the fluid is augmented. Figure 15 shows that when Nb increases, $\theta(\eta)$ decreases near the sheet and takes on an inverse nature far away from it. In reality, a larger Nb causes more Brownian diffusion with lesser viscous forces, and therefore, a hike in the temperature profile is observed. $\phi(\eta)$ is enhanced near the sheet for uplifting Nb values, while a reverse influence is seen away from the sheet, as shown in Figure 16. According to Figure 17, with upsurging values of Nt, $\theta(\eta)$ is increased. Physically, an increase in Nt causes a stronger thermophoretic force, which enriches the fluid's temperature. Figure 18 shows that $\phi(\eta)$ decreases towards the sheet, while the opposite trend is seen further away from the sheet in terms of Nt.

Figure 19 shows that an improvement in K_1 leads to a significant fall in $\phi(\eta)$. A devastating chemical reaction corresponds to a positive K_1. As a result, an improvement in K_1 causes a decrease in species concentration. In Figure 20, it is seen that for growing values of Le, $\phi(\eta)$ is reduced. Lewis number is basically the relation between thermal diffusivity to mass diffusivity. So, higher Lewis number implies less mass diffusion in the fluid flow. Hence, species concentration is lessened. Figure 21 reveals that there is an upward

Table 3: Numerical values of the skin friction coefficients when $\zeta_C = \zeta_T = 0.4$, $L = \lambda = N = n = 0.5$, $We_1 = We_2 = 1.6$ and $\beta_1 = 0.6$

λ	M	S	λ_1	λ_2	α_1	α_2	$-\sqrt{Re_x}C_{fx}$	$-\sqrt{Re_y}C_{fy}$
0.4	1	0.2	0.3	0.3	0.4	0.4	0.953465	1.833773
1							0.921040	1.766040
2							0.871657	1.663539
0.4	2						1.095809	2.137023
	3						1.204741	2.373294
	1	0.6					1.006186	1.945268
		1.0					1.052698	2.044400
		0.2	0.6				0.952571	–
			0.9				0.951679	–
			0.3	0.6			0.953415	–
				0.9			0.953364	–
				0.3	0.7		0.725064	–
					1.0		0.586788	–
					0.4	0.7	–	1.812118
						1.0	–	1.799073

trend in $\phi(\eta)$ with the progress of the parameter E. Boosted E values aid in the speeding up of chemical reactions and hence the species concentration is escalated.

The numerical values of the local skin-friction coefficients for various values of the controlling parameters λ, M, S, λ_1, λ_2, α_1 and α_2 are set forth in Table 3. For higher values of M and S, both $\sqrt{Re_x}C_{fx}$ and $\sqrt{Re_y}C_{fy}$ are increased whereas reverse trend is detected w.r.t. λ, λ_1, λ_2, α_1 and α_2. Local Nusselt and Sherwood numbers calculated for flow parameters M, Rd, Bi, Nb, Nt, K_1, Le and E are described in Table 4. Increasing trend of $\frac{Nu_x}{\sqrt{Re_x}}$ is found for Rd and Bi but opposite nature is noticed for M, Nb and Nt. Growing values of Nt and E imply increasing tendency of $\frac{Sh_x}{\sqrt{Re_x}}$ whereas converse behavior is found w.r.t. Nb, Le and K_1.

6 Conclusions

The present analysis explores the aspects of nonlinear thermal radiation and activation energy on unsteady convective heat and mass transport phenomena of Williamson nanofluid over a stretching sheet in the existence of Lorentz force and chemical reaction. Moreover, Navier's velocity slip and convective heating conditions are imposed at the surface boundary. The following are some of the significant outcomes from the simulation of the problem:

- A diminishing nature is observed for the velocity profiles with the improvement in unsteadiness and the intensity of the Lorentz force.

Table 4: Numerical values of the local Nusselt and Sherwood numbers when $Pr = 1.4$, $\theta_w = 1.1$ and $Le = 1.5$

M	Rd	Bi	Nb	Nt	K_1	Le	E	$\frac{Nu_x}{\sqrt{Re_x}}$	$-\frac{Sh_x}{\sqrt{Re_x}}$
1	0.1	0.3	0.4	0.5	0.3	1.0	0.6	0.267377	0.300448
2								0.262851	–
3								0.259108	–
1	0.5							0.361351	–
	1.0							0.474500	–
	0.1	0.5						0.395575	–
		0.7						0.497861	–
		0.3	0.6					0.267364	0.200288
			0.8					0.267357	0.150212
			0.4	1.0				0.267028	0.600074
				1.5				0.266675	0.898864
				0.5	0.6			–	0.300408
					0.8			–	0.300385
					0.3	0.5		–	0.300598
						1.5		–	0.300337
						1.0	2.0	–	0.300487
							5.0	–	0.300502

- The temperature distribution is enhanced as the thermal radiation and the convective heating at the bottom of the surface is boosted.
- The thermophoretic force and the activation energy are found to have strong influence on rising the species concentration far away from the sheet. However, the impact is getting reversed near the sheet.
- The skin friction coefficients are uplifted with the increase of unsteadiness and the magnetic impact.
- There is an enhancement in heat transfer rate at the surface for growing value of Biot number and thermal radiation parameter.
- Rate of mass transfer at the wall is improved as the values of thermophoresis and the activation energy parameters increase.

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Markovian Queueing Model with Single Working Vacation and Server Breakdown

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In this article, we investigate a Markovian queuing model with server breakdown and Single working vacation. Arrival follows Poisson process with parameter λ. Service while the single working vacation epoch, normal working epoch together with vacation epoch are all exponentially assigned with rate μ_b, μ_v and ηrespectively. After taking first vacation the server wait idle in the system to serve. This type of vacation is called Single Working Vacation (SWV). If the queue length increases, service rate changes from slow rate to normal rate. When the server may subject to sudden breakdown with rate α and after it should be repaired and goes to normal service with rate β. This queue model has been analysed with the help of Matrix Geometric Method (MGM) to find steady state probability vectors. Using it some performance measure is also determined.

AMS subject classification number— 60K25,60K30 and 90B22
Key Words —- Working Vacation (WV), Single Working Vacation (SWV) , Stability condition, Server breakdown, Matrix Geometric Method (MGM);

1 Introduction

Queueing systems with SWV and absolute service have acquired importance over the most recent twenty years because of large extent uses mainly manufacturing system, service system, telecommunications, and computer system. Its discoveries might be utilized to give quicker client support, increment traffic stream, further develop request shipments from a stockroom, or plan information organizations and call focuses. Numerous significant utilization of the

queueing theory are traffic flow vehicles, airplane, individuals, correspondences, booking patients in medical clinics, occupations on machines, programs on PC, and office configuration banks, mail depots, stores. In numerous genuine queueing circumstances, after assistance fulfillment, if no customer in the queue, a server goes on a vacation epoch. This type is known as vacation queue. Using survey paper by Doshi (1986) many researchers introduced queueing model with vacations.

Servi and Finn (2002) developed an M/M/1 queueing model upon WV. Wu and Takagi (2003) analysed an M/G/1 queueing model upon MWV. Analysis of GI/M/1 queueing model upon MWV studied by Baba (2005). Server will only take one WV if the queue become null. So, if the queue has no customers when the server come back from SWV, he will idle on the system and wait for the customers to arrive instead of picking up another WV. A multi-server system with an SWV were proposed by Lin and Ke (2009). The use of inactive epoch a M/G/1 model were studied by Levy and Yechiali (1975).

The vacation models and the model in which the server may goes to breakdowns and repairs, is well ascertained in survey papers B.T. Doshi. William J. Gray et al (2000) overworked on multiple types of server breakdowns in queueing theory. A queueing model with server breakdowns, repairs, vacations, and backup server is studied by Srinivas R.Chakravarthy (2020).Agelenbe et al (1991) analyzed the queues with negative arrivals. A matrix-Geometric method approach is a useful tool for solving the more complex queueing problems. Neuts (1978) deliberate Markov c hains with applications queueing theory, which have a matrix geometric invariant probability vector. Neuts (1981) derived Matrix Geometric solution in stochastic models.

Transient solution for the queue-size distribution in a finite-buffer model with general independent input stream and single working vacation was explained by Wojciech M Kempa et al (2018). Rachita Sethi et al (2019) studied performance analysis of machine repair problem with working vacation and service interruptions. Seenivasan et al (2021) studied performance examination of two heterogeneous servers queuing model with an irregularly reachable server utilizing MGM. Seenivasan et al (2021) investigated a retrial queueing model with two heterogeneous servers using the MGM. Praveen Deora et al (2021) analyzed the cost analysis and optimization of machine repair model with working vacation and feedback policy. M/M/1 queueing model with working vacation and two type of server breakdown was discussed by Praveen Kumar Agrawal et al (2021).

[1] Our study, deals with an SWV and server breakdown in M/M/1 queueing model. In accordance with FCFS principle customers are served. This model has been analysed using MGM. The excess of this study designated as follows. We providing construction of model in section 2. Performance measures formalized in section 3. Mathematical illustrations solved in section 4. And brief conclusion in final section.

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2 Construction of the Model

We consider a Single Working Vacation (SWV) and Server breakdown in M/M/1 model. The customers show up in line as per Poisson process with parameter λ. They create a queue dependent on her/his request for appearance. At a normal working period, influx customers served at a service rate μ_b, following an exponential distribution. The server starts a single vacation of arbitrary length if there is no customer in the system with parameter η follows an exponential distribution. During a SWV period, influx customers get service with rate μ_v, following an exponential distribution. If the queue forms, then the server chop and shift its rate from μ_v to μ_b, the normal working interval starts. For next vacation server waits idle to serve influx customers. This type of vacation is called Single Working Vacation (SWV). If not, the server starts a normal working period when a customer arrival occurs.

When the server may subject to sudden breakdown with rate α and after it should be repaired and goes to normal service with rate β. The transition rate diagram is shown in Figure 1.

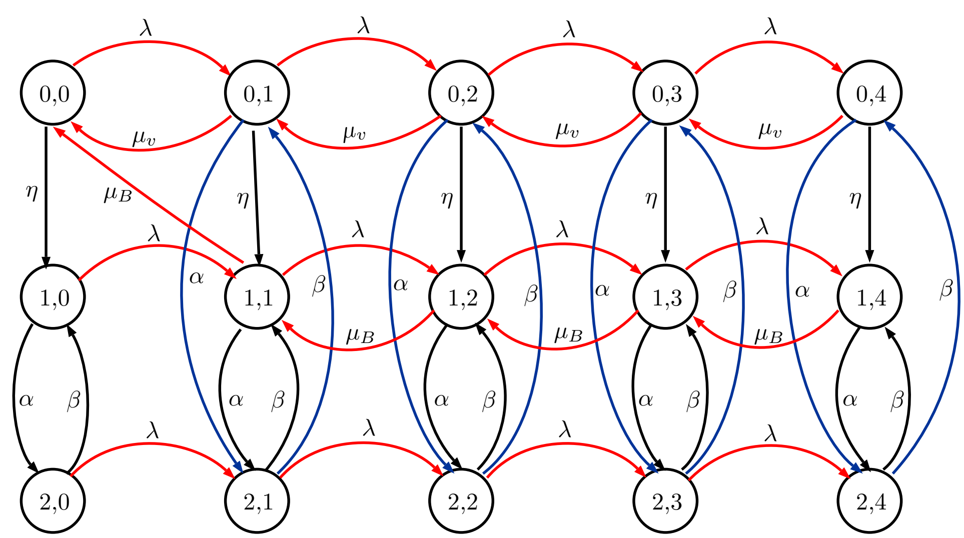

Figure 1: The transition rate diagram

Let $\{k(t), n(t) : t \geq 0\}; \lim_{t\to\infty} p\{k(t) = i, n(t) = j\}$ be a Markov process, where $k(t)$ and $n(t)$ represent state of process at time t respectively.
$k(t) = 0$, when server is on SWV,
$k(t) = 1$, when server is on normal working epoch
$k(t) = 2$, when server is on breakdown
n(t) denotes total customer in the system.
The Quasi-birth and death Process along with the state space Ω as follow

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$\Omega = \{(0,0)U(1,0)U(2,0)U(i,j); i = 0,1,2, j = 1,2,....,n \geq 1\}$

Consider a QBD process with Infinitesimal generator matrix Q is presented below

$$Q = \begin{pmatrix} A_0 & C_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ C_0 & C_2 & C_3 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & C_3 & C_2 & C_3 & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & C_3 & C_2 & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & C_3 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Where

$$A_0 = \begin{pmatrix} -(\lambda+\eta) & \eta & 0 \\ 0 & -(\lambda+\alpha) & \alpha \\ 0 & \beta & -(\beta+\lambda) \end{pmatrix}; C_1 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix};$$

$$C_0 = \begin{pmatrix} \mu_v & 0 & 0 \\ \mu_b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; C_2 = \begin{pmatrix} -(\lambda+\mu_v+\alpha+\eta) & \eta & \alpha \\ 0 & -(\lambda+\mu_b+\alpha) & \alpha \\ \beta & \beta & -(2\beta+\lambda) \end{pmatrix};$$

$$C_3 = \begin{pmatrix} \mu_v & 0 & 0 \\ 0 & \mu_b & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We define $p_{ij} = \{k = i, n = j\}$; where j denote number of customers in the system & i denote the server state.

Probability vector are defined as $P = (P_0, P_1, P_2, \ldots)$ where, $P_j = (p_{0j}, p_{1j}, p_{2j})$, $j = 0,1,2,3,...$

The static probability row matrix is represented by using $PQ = O$. (1)

$P_j = P_0 R^j$ where $j \geq 1$ (2)

The normalizing equation is defined by

$P_0 [I - R]^{-1} e = 1$ (3)

Where 'e' is the column unit vector with all its element equal to one.

The static condition of such a QBD, (See Neuts (1981)) can be obtained by the drift condition

$PC_1 e < PC_3 e$ (4)

Where the row vector $P = (P_0, P_1, P_2)$ is obtained from the Infinitesimal generator

$S = C_1 + C_2 + C_3$. S is given by

$$S = \begin{pmatrix} -(\alpha+\eta) & \eta & \alpha \\ 0 & -\alpha & \alpha \\ \beta & \beta & 2\beta \end{pmatrix} \quad (5)$$

S is irreducible and the row vector P can be shown to be unique such that

$PS = 0$ and $Pe = 1$ (6)

From Equation(6), we have

$P_1 = (\frac{2\eta-\alpha}{\alpha})P_0$

$P_2 = (\frac{\eta+\alpha}{\beta})P_0$

$P_0 = [1 + (\frac{\eta+\alpha}{\beta}) + (\frac{2\eta-\alpha}{\alpha})]^{-1}$ (7)

The static condition takes format

$\lambda[P_0 + P_1 + P_2] < \mu_b P_0 + \mu_v P_1$ (8)

Equation(5) gives the static probability of S.

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Once the rate matrix R obtained, the probability vectors P_j's $(j \geq 1)$ are obtained from Eq.(2) and Eq.(3).

3 Performance Measures

Performance measure have been found using steady-state probabilities as given below

When the server is idle mean no. of customers presented $E(I) = P_0$ (9)

When the server is SWV mean no. of customers presented

$$E(SWV) = \sum_{j=0}^{\infty} jp_{0j} \tag{10}$$

When the server is normal busy period mean no. of customers presented

$$E(BP) = \sum_{j=1}^{\infty} jp_{1j} \tag{11}$$

When the server is on breakdown mean no. of customers presented

$$E(BD) = \sum_{j=1}^{\infty} jp_{1j} \tag{12}$$

Mean no. of customer in the system is

$$E(N) = E(I) + E(WV) + E(BP) + E(BD) \tag{13}$$

4 Mathematical Study

Here, we make mathematical calculation for model given by the segment above. Our goals are to show effect of a parameter on system features. By modifying λ, four illustrations are presented in these sections.

The parameter λ value varies and all other argument values are fixed. Illustration 1 to Illustration 4 is presented below.

Illustration 1

We take $\lambda = 0.1$,$\mu_b = 0.6$,$\mu_v = 0.5$,$\alpha = 0.2$,$\eta = 0.5$,$\beta = 0.7$ and the rate matrix is

$$R = \begin{pmatrix} 0.0942 & 0.0781 & 0.0193 \\ 0.0126 & 0.1347 & 0.0167 \\ 0.0534 & 0.1106 & 0.0834 \end{pmatrix}$$

Table 1.Probability vectors

	p_{0j}	p_{1j}	p_{2j}	Total
P_0	0.1215	0.5476	0.1479	0.8170
P_1	0.0262	0.0996	0.0232	0.1496
P_2	0.0050	0.0181	0.0042	0.0273
P_3	0.0009	0.0033	0.0007	0.0049
P_4	0.0002	0.0006	0.0001	0.0009
P_5	0.0000	0.0001	0.0000	0.0001
Total				0.9998

By substituting R matrix in equation (1) vector P_0 are obtained and normalization equation $P_0\,[I-R]^{-1}\,e = 1$ for the mathematical argument selected previously, row vector P_1 is granted by $P_0 = (0.1215, 0.05476, 0.1479)$. More, the balance vector P_j's gained from $P_j = P_0R^j, j = 1, 2, 3, \ldots$ and are shown in Table 1. Column 2, 3 and 4 contains the three elements of $P_j, j = 0, 1, 2, \ldots$. Final column constitutes the total of two elements. Total probability was confirmed to be $0.9998 \approx 1$.

Illustration 2

We take $\lambda = 0.2$,$\mu_b = 0.6$,$\mu_v = 0.5$,$\alpha = 0.2$,$\eta = 0.5$,$\beta = 0.7$ and the rate matrix is

$$R = \begin{pmatrix} 0.1807 & 0.1644 & 0.0313 \\ 0.0250 & 0.2694 & 0.0274 \\ 0.1024 & 0.2328 & 0.1507 \end{pmatrix}$$

Table 2.Probability vectors

	p_{0j}	p_{1j}	p_{2j}	Total
P_0	0.1655	0.3782	0.0979	0.6416
P_1	0.0494	0.1519	0.0303	0.2316
P_2	0.0158	0.0561	0.0103	0.0822
P_3	0.0053	0.0201	0.0036	0.0290
P_4	0.0018	0.0070	0.0013	0.0101
P_5	0.0006	0.0025	0.0004	0.0035
P_6	0.0002	0.0009	0.0002	0.0013
P_7	0.0001	0.0003	0.0001	0.0005
P_8	0.0000	0.0001	0.0000	0.0001
Total				0.9999

By substituting R matrix in equation (1) vector P_0 are obtained and normalization equation $P_0\,[I-R]^{-1}\,e = 1$ for the mathematical argument selected previously, row vector P_1 is granted by $P_0 = (0.1655, 0.3782, 0.0979)$. More, the

balance vector P_j's gained from $P_j = P_0 R^j, j = 1, 2, 3, \ldots$ and are shown in Table 1. Column 2, 3 and 4 contains the three elements of $P_j, j = 0, 1, 2, \ldots$. Final column constitutes the total of two elements. Total probability was confirmed to be $0.9999 \approx 1$.

Illustration 3

We take $\lambda = 0.3$,$\mu_b = 0.6$,$\mu_v = 0.5$,$\alpha = 0.2$,$\eta = 0.5$,$\beta = 0.7$ and the rate matrix is

$$R = \begin{pmatrix} 0.2587 & 0.2552 & 0.0391 \\ 0.0363 & 0.4018 & 0.0345 \\ 0.1452 & 0.3612 & 0.2067 \end{pmatrix}$$

Table 3.Probability vectors

	p_{0j}	p_{1j}	p_{2j}	Total
P_0	0.1615	0.2511	0.0640	0.4766
P_1	0.0602	0.1612	0.0282	0.2536
P_2	0.0257	0.0919	0.0139	0.1315
P_3	0.0120	0.0485	0.0070	0.0675
P_4	0.0059	0.0251	0.0036	0.0346
P_5	0.0030	0.0129	0.0018	0.0177
P_6	0.0015	0.0066	0.0009	0.0090
P_7	0.0008	0.0034	0.0005	0.0047
P_8	0.0004	0.0017	0.0002	0.0023
P_9	0.0002	0.0009	0.0001	0.0012
P_{10}	0.0001	0.0005	0.0001	0.0007
P_{11}	0.0001	0.0002	0.0000	0.0003
P_{12}	0.0000	0.0001	0.0000	0.0001
Total				0.9997

By substituting R matrix in equation (1) vector P_0 are obtained and normalization equation $P_0 [I - R]^{-1} e = 1$ for the mathematical argument selected previously, row vector P_1 is granted by $P_0 = (0.1615, 0.2511, 0.0640)$. More, the balance vector P_j's gained from $P_j = P_0 R^j, j = 1, 2, 3, \ldots$ and are shown in Table 1. Column 2, 3 and 4 contains the three elements of $P_j, j = 0, 1, 2, \ldots$. Final column constitutes the total of two elements. Total probability was confirmed to be $0.9997 \approx 1$.

Illustration 4

We take $\lambda = 0.4$,$\mu_b = 0.6$,$\mu_v = 0.5$,$\alpha = 0.2$,$\eta = 0.5$,$\beta = 0.7$ and the rate matrix is

$$R = \begin{pmatrix} 0.3275 & 0.3420 & 0.0441 \\ 0.0453 & 0.5253 & 0.0392 \\ 0.1802 & 0.4836 & 0.2546 \end{pmatrix}$$

Table 4.Probability vectors

	p_{0j}	p_{1j}	p_{2j}	Total
P_0	0.1304	0.1578	0.0421	0.3303
P_1	0.0574	0.1478	0.0227	0.2279
P_2	0.0296	0.1083	0.0141	0.1520
P_3	0.0171	0.0738	0.0091	0.1000
P_4	0.0106	0.0491	0.0060	0.0657
P_5	0.0068	0.0323	0.0039	0.0430
P_6	0.0044	0.0212	0.0026	0.0282
P_7	0.0029	0.0139	0.0017	0.0185
P_8	0.0019	0.0091	0.0011	0.0121
P_9	0.0012	0.0059	0.0007	0.0078
P_{10}	0.0008	0.0039	0.0005	0.0052
P_{11}	0.0005	0.0025	0.0003	0.0033
P_{12}	0.0003	0.0017	0.0002	0.0022
P_{13}	0.0003	0.0011	0.0001	0.0014
P_{14}	0.0001	0.0007	0.0001	0.0009
P_{15}	0.0001	0.0004	0.0001	0.0006
P_{16}	0.0001	0.0003	0.0000	0.0004
P_{17}	0.0000	0.0002	0.0000	0.0002
P_{18}	0.0000	0.0001	0.0000	0.0001
Total				0.9998

By substituting R matrix in equation (1) vector P_0 are obtained and normalization equation $P_0\left[I-R\right]^{-1}e = 1$ for the mathematical argument selected previously, row vector P_1 is granted by $P_0 = (0.1304, 0.1578, 0.0421)$. More, the balance vector P_j's gained from $P_j = P_0R^j, j = 1, 2, 3, \ldots$ and are shown in Table 1. Column 2, 3 and 4 contains the three elements of $P_j, j = 0, 1, 2, \ldots$. Final column constitutes the total of two elements. Total probability was confirmed to be $0.9998 \approx 1$.

Table 4.Performance Measures

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λ	$E(I)$	$E(SWV)$	$E(BP)$	$E(BD)$	$E(N)$
0.1	0.8170	0.0397	0.1486	0.0347	1.0400
0.2	0.6416	0.1090	0.3736	0.0708	1.1950
0.3	0.4766	0.2079	0.7529	0.1128	1.5502
0.4	0.3303	0.3412	1.4211	0.1792	2.2718

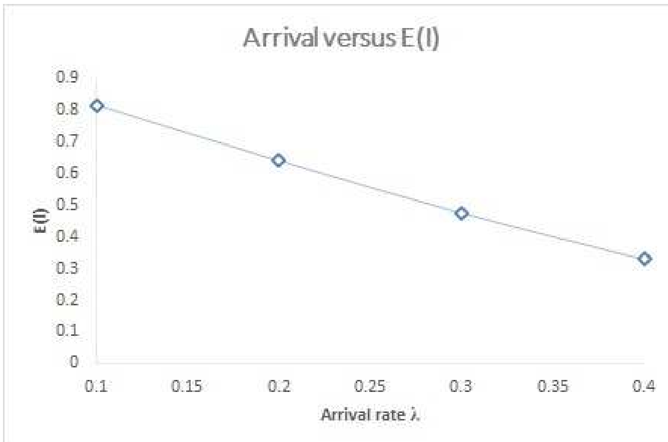

Figure 2

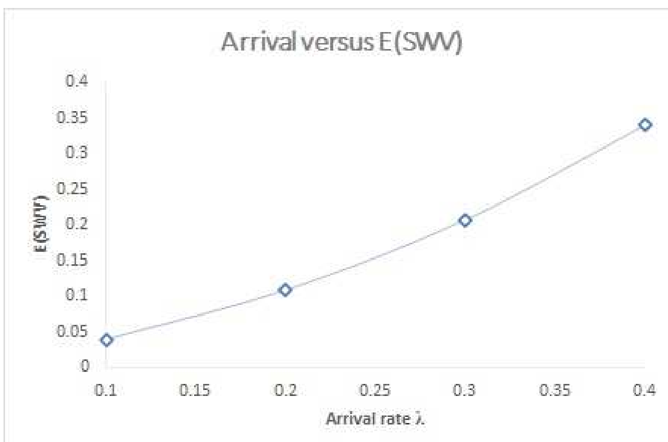

Figure 3

9

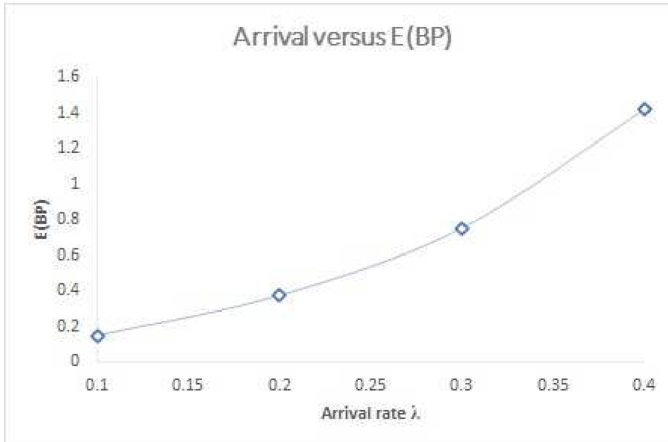

Figure 4

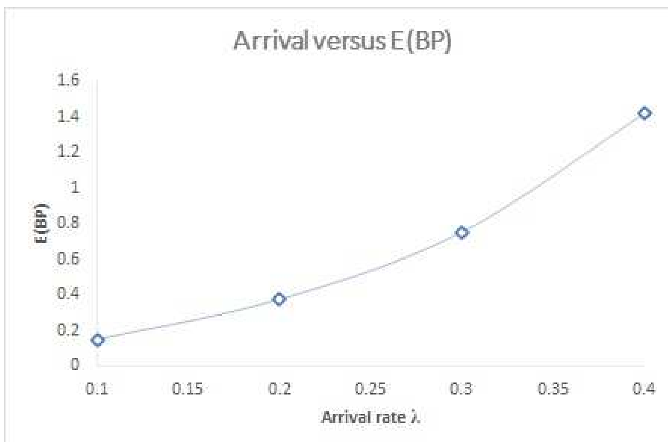

Figure 5

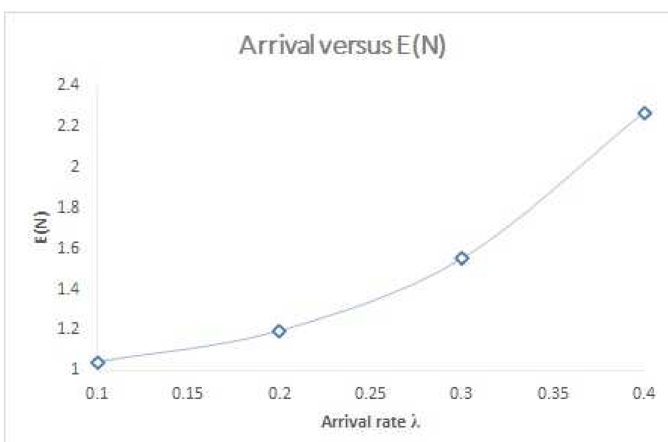

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Figure 6

Out of the above figures, we derived few performance measurements with the effect of λ such as mean no., of customer if server is idle, mean no., of customer if server is on SWV, mean no., of customer if server is on busy period, mean no., of customer if server is on breakdown and mean no., of customers throughout system respectively. From Figure 2 shows that arrival rate increases, mean no., of customer if server is idle decreases, Figure 3, Figure 4, Figure 5 and Figure 6 shows arrival rate increases, mean no., of customer if server is SWV, busy period, breakdown and mean no., of customers throughout system increases.

5 CONCLUSION

In this article, we have studied a single-server queueing model along with SWV and server breakdown. We derived the static probability row vector by MGM and also we derived some performance measures for mean no., of customers in the system during server is idle, SWV, normal busy period, breakdown and mean no., of customers throughout system respectively with the effect of λ.

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Generalized fractional operators and their image formulas

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Abstract

Numerous image formulae for a diversity of polynomials and functions subjected to a variety of fractional integrals and derivatives have been given. In this paper, we aimto construct image formulae for the product of incomplete H-functions and a general class of polynomials under the Katugampola fractional integral and derivative operators . We also provide some particular instances of our main findings in corollaries, among many others.

Keywords: Fractional Integral operators, Incomplete H-functions, Fox's H-function, Mellin-Barnes type contour integral.
Mathematics Subject Classification 2010: 26A33; 33C60; 44A10; 34A08; 33B20.

1 Introduction and preliminaries

We begin by recalling the well-known Gamma function Γ defined by (see, e.g., [19, Section 1.1])

$$\Gamma(\mu) = \begin{cases} \int_0^\infty e^{-\mathfrak{v}}\mathfrak{v}^{\mu-1}d\mathfrak{v} & (\Re(\mu) > 0) \\ \frac{\Gamma(\mu+k)}{(\mu)_k} & (\mu \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0};\ k \in \mathbb{N}_0), \end{cases} \tag{1.1}$$

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where the Pochhammer symbol $(\mu)_\nu$ $(\mu,\ \nu \in \mathbb{C})$ is defined, in terms of Gamma function Γ (see, e.g., [19, p. 2 and p. 5]), by

$$\begin{aligned}(\mu)_\nu &= \frac{\Gamma(\mu+\nu)}{\Gamma(\mu)} \quad (\mu+\nu \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0},\ \nu\in\mathbb{C}\setminus\{0\};\ \mu\in\mathbb{C}\setminus\mathbb{Z}_{\leq 0},\ \nu=0)\\ &= \begin{cases} 1 & (\nu=0,\ \mu\in\mathbb{C}\setminus\{0\}),\\ \mu(\mu+1)\cdots(\mu+n-1) & (\nu=n\in\mathbb{N},\ \mu\in\mathbb{C}),\end{cases}\end{aligned} \tag{1.2}$$

it being assumed that $(0)_0 = 1$. Here and throughout, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$, and $\mathbb{N}$ denote the sets of complex numbers, real numbers, positive real numbers, integers, and positive integers, respectively. Also let $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$ and $\mathbb{Z}_{\leq \ell}$ be the set of integers which are less than or equal to some integer $\ell \in \mathbb{Z}$. The incomplete Gamma function $\gamma(\mu, u)$ and its complement $\Gamma(\mu, u)$ defined by

$$\gamma(\mu,u) = \int_0^u e^{-\mathfrak{v}}\mathfrak{v}^{\mu-1}d\mathfrak{v} \qquad (u \geq 0;\ \Re(\mu) > 0), \tag{1.3}$$

and

$$\Gamma(\mu,u) = \int_u^\infty e^{-\mathfrak{v}}\mathfrak{v}^{\mu-1}d\mathfrak{v} \qquad (u \geq 0;\ \Re(\mu) > 0 \text{ when } u = 0), \tag{1.4}$$

respectively, satisfy the following relation:

$$\gamma(\mu,u) + \Gamma(\mu,u) = \Gamma(\mu) \qquad (\Re(\mu) > 0). \tag{1.5}$$

Srivastava et al. [21] used the incomplete Gamma functions to introduce the following incomplete H-functions (see also [5]):

$$\begin{aligned}\gamma^{\mathsf{m},\mathsf{n}}_{\mathsf{p},\mathsf{q}}(z) &= \gamma^{\mathsf{m},\mathsf{n}}_{\mathsf{p},\mathsf{q}}\left[\ z\ \middle|\ \begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_i,\mathsf{E}_i)_{2,\mathsf{p}}\\ \\ (\mathsf{f}_i,\mathsf{F}_i)_{1,\mathsf{q}}\end{matrix}\right]\\ &= \gamma^{\mathsf{m},\mathsf{n}}_{\mathsf{p},\mathsf{q}}\left[\ z\ \middle|\ \begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_2,\mathsf{E}_2),\cdots,(\mathsf{e}_\mathsf{p},\mathsf{E}_\mathsf{p})\\ \\ (\mathsf{f}_1,\mathsf{F}_1),(\mathsf{f}_2,\mathsf{F}_2),\cdots,(\mathsf{f}_\mathsf{q},\mathsf{F}_\mathsf{q})\end{matrix}\right]\\ &:= \frac{1}{2\pi \mathrm{i}}\int_{\mathfrak{C}} \mathbb{G}(\xi,\mathsf{y})\ z^{-\xi}d\xi,\end{aligned} \tag{1.6}$$

where

$$\mathbb{G}(\xi,\mathsf{y}) = \frac{\gamma(1-\mathsf{e}_1-\mathsf{E}_1\xi,\mathsf{y})\prod\limits_{i=1}^{\mathsf{m}}\Gamma(\mathsf{f}_i+\mathsf{F}_i\xi)\prod\limits_{i=2}^{\mathsf{n}}\Gamma(1-\mathsf{e}_i-\mathsf{E}_i\xi)}{\prod\limits_{i=\mathsf{m}+1}^{\mathsf{q}}\Gamma(1-\mathsf{f}_i-\mathsf{F}_i\xi)\prod\limits_{i=\mathsf{n}+1}^{\mathsf{p}}\Gamma(\mathsf{e}_i+\mathsf{E}_i\xi)}; \tag{1.7}$$

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$$\begin{aligned}\Gamma_{\mathsf{p},\mathsf{q}}^{\mathsf{m},\mathsf{n}}(z) &= \Gamma_{\mathsf{p},\mathsf{q}}^{\mathsf{m},\mathsf{n}}\left[\, z \;\middle|\; \begin{matrix} (\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_i,\mathsf{E}_i)_{2,\mathsf{p}} \\ \\ (\mathsf{f}_i,\mathsf{F}_i)_{1,\mathsf{q}} \end{matrix} \right] \\ &= \Gamma_{\mathsf{p},\mathsf{q}}^{\mathsf{m},\mathsf{n}}\left[\, z \;\middle|\; \begin{matrix} (\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_2,\mathsf{E}_2),\cdots,(\mathsf{e}_\mathsf{p},\mathsf{E}_\mathsf{p}) \\ \\ (\mathsf{f}_1,\mathsf{F}_1),(\mathsf{f}_2,\mathsf{F}_2),\cdots,(\mathsf{f}_\mathsf{q},\mathsf{F}_\mathsf{q}) \end{matrix} \right] \\ &:= \frac{1}{2\pi\mathrm{i}} \int_{\mathfrak{C}} \mathbb{F}(\xi,\mathsf{y})\; z^{-\xi} d\xi, \end{aligned} \tag{1.8}$$

where

$$\mathbb{F}(\xi,\mathsf{y}) = \frac{\Gamma(1-\mathsf{e}_1-\mathsf{E}_1\xi,\mathsf{y}) \prod\limits_{i=1}^{\mathsf{m}} \Gamma(\mathsf{f}_i+\mathsf{F}_i\xi) \prod\limits_{i=2}^{\mathsf{n}} \Gamma(1-\mathsf{e}_i-\mathsf{E}_i\xi)}{\prod\limits_{i=\mathsf{m}+1}^{\mathsf{q}} \Gamma(1-\mathsf{f}_i-\mathsf{F}_i\xi) \prod\limits_{i=\mathsf{n}+1}^{\mathsf{p}} \Gamma(\mathsf{e}_i+\mathsf{E}_i\xi)}. \tag{1.9}$$

For convergence conditions of these incomplete H-functions as well as the description of the contour $\mathfrak{C}$, one may refer to [21]. They [21] explored a variety of intriguing properties of these incomplete H-functions, such as decomposition and reduction formulas, derivative formulas, various integral transforms, and computational representations, as well as applied some significantly general RiemannLiouville and Weyl type fractional integral operators to each of these incomplete H-functions.

Srivastava [17] introduced the following general class of polynomials:

$$S_{\mathsf{n}}^{\mathsf{m}}[x] = \sum_{k=0}^{[\mathsf{n}/\mathsf{m}]} \frac{(-\mathsf{n})_{\mathsf{m}k}}{k!} A_{\mathsf{n},k}\, x^k \qquad (\mathsf{m}\in\mathbb{N},\ \mathsf{n}\in\mathbb{N}_0), \tag{1.10}$$

where the coefficients $A_{\mathsf{n},k}$ (n, $k \in \mathbb{N}_0$) are arbitrary real or complex constants. By properly specializing $A_{\mathsf{n},k}$, the general class of polynomials may generate many existing polynomials as special instances, including Jacobi and Laguerre polynomials (see, e.g., [15]). In particular, setting $A_{0,0} = 1$ and $x = 0$ reduces $S_{\mathsf{n}}^{\mathsf{m}}[x]$ to unity.

There have been many introductions and investigations of fractional integrals and derivatives. Two of them are recalled here. The left-sided and right-sided Riemann-Liouville fractional integrals $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha \in \mathbb{C}$ are defined as (see, e.g., [10, 12–14])

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau)\, d\tau \quad (x > a,\ \Re(\alpha) > 0), \tag{1.11}$$

and

$$\left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau-x)^{\alpha-1} f(\tau)\, d\tau \quad (b > x,\ \Re(\alpha) > 0), \tag{1.12}$$

respectively. The Riemann-Liouville fractional derivatives $D^{\alpha}_{a+}f$ and $D^{\alpha}_{b-}f$ of order $\alpha \in \mathbb{C}$ ($\Re(\alpha) \geq 0$) are defined by

$$(D^{\alpha}_{a+}f)(x) = \left(\frac{d}{dx}\right)^{\mathsf{n}} \left(I^{\mathsf{n}-\alpha}_{a+}f\right)(x) \quad (x > a) \tag{1.13}$$

and

$$(D^{\alpha}_{b-}f)(x) = \left(-\frac{d}{dx}\right)^{\mathsf{n}} \left(I^{\mathsf{n}-\alpha}_{b-}f\right)(x) \quad (x < b), \tag{1.14}$$

where $\mathsf{n} = [\Re(\alpha)] + 1$.

For $\rho \in \mathbb{R} \setminus \{0\}$, the left-sided and right-sided Katugampola fractional integrals, respectively, denoted by ${}^{\rho}I^{\lambda}_{a+}$ and ${}^{\rho}I^{\lambda}_{b-}$ of order $\lambda \in \mathbb{C}$ ($\Re(\lambda) > 0$), are defined as (see [7])

$$\left({}^{\rho}I^{\lambda}_{a+}\phi\right)(s) = \frac{\rho^{1-\lambda}}{\Gamma(\lambda)} \int_a^s \frac{\tau^{\rho-1}\phi(\tau)}{(s^{\rho} - \tau^{\rho})^{1-\lambda}} d\tau \quad (s > a), \tag{1.15}$$

and

$$\left({}^{\rho}I^{\lambda}_{b-}\phi\right)(s) = \frac{\rho^{1-\lambda}}{\Gamma(\lambda)} \int_s^b \frac{\tau^{\rho-1}\phi(\tau)}{(\tau^{\rho} - s^{\rho})^{1-\lambda}} d\tau \quad (b > s). \tag{1.16}$$

It is noted that

(i) when $\rho = 1$, (1.15) and (1.16), respectively, reduce to Riemann-Liouville fractional integrals (1.11) and (1.12);

(ii) taking $\rho \to 0^+$, (1.15) and (1.16), respectively, reduce to the famous Hadamard fractional integrals (see [6]; see also [7]):

$$\left(H^{\lambda}_{a+}\phi\right)(s) = \frac{1}{\Gamma(\lambda)} \int_a^s \left(\log\frac{s}{\tau}\right)^{\lambda-1} \frac{\phi(\tau)}{\tau}\, d\tau \quad (s > a,\, \Re(\lambda) > 0), \tag{1.17}$$

and

$$\left(H^{\lambda}_{b-}\phi\right)(s) = \frac{1}{\Gamma(\lambda)} \int_s^b \left(\log\frac{\tau}{s}\right)^{\lambda-1} \frac{\phi(\tau)}{\tau}\, d\tau \quad (s < b,\, \Re(\lambda) > 0). \tag{1.18}$$

The matching Katugampola fractional derivatives on the left and right sides, designated respectively by ${}^{\rho}D^{\lambda}_{a+}$ and ${}^{\rho}D^{\lambda}_{b-}$, are defined as (see [8])

$$\begin{aligned}\left({}^{\rho}D^{\lambda}_{a+}\phi\right)(s) &= \left(s^{1-\rho}\frac{d}{ds}\right)^{\mathsf{n}} \left({}^{\rho}I^{\mathsf{n}-\lambda}_{a+}\phi\right)(s) \\ &= \frac{\rho^{\lambda-\mathsf{n}+1}}{\Gamma(\mathsf{n}-\lambda)} \left(s^{1-\rho}\frac{d}{ds}\right)^{\mathsf{n}} \int_a^s \frac{\tau^{\rho-1}\phi(\tau)}{(s^{\rho} - \tau^{\rho})^{\lambda-\mathsf{n}+1}} d\tau, \end{aligned} \tag{1.19}$$

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and

$$\begin{aligned}\left({}^{\rho}D_{b-}^{\lambda}\phi\right)(s) &= \left(-s^{1-\rho}\frac{d}{ds}\right)^{\mathsf{n}}\left({}^{\rho}I_{b-}^{\mathsf{n}-\lambda}\phi\right)(s)\\ &= \frac{\rho^{\lambda-\mathsf{n}+1}}{\Gamma(\mathsf{n}-\lambda)}\left(-s^{1-\rho}\frac{d}{ds}\right)^{\mathsf{n}}\int_s^b \frac{\tau^{\rho-1}\phi(\tau)}{(\tau^{\rho}-s^{\rho})^{\lambda-\mathsf{n}+1}}\,d\tau, \end{aligned} \tag{1.20}$$

where $\mathsf{n} = [\Re(\lambda)] + 1$.

The identities in Lemmas 1.1 and 1.2 provide the image formulae for the power function t^{α} when the fractional integral and derivative operators of Katugampola are used. In this case, we make major use of the well-known beta function (see, e.g., [19, p. 8]):

$$B(\alpha,\,\beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t & (\Re(\alpha)>0,\ \Re(\beta)>0)\\ \displaystyle\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\,\beta\in\mathbb{C}\setminus\mathbb{Z}_{\leq 0}). \end{cases} \tag{1.21}$$

Proofs have been omitted.

Lemma 1.1 *Let $\rho > 0$, $\Re(\alpha) > 0$, and $\Re(\lambda) > 0$. Then*

$$\left({}^{\rho}I_{0+}^{\lambda}t^{\alpha}\right)(s) = \rho^{-\lambda}\frac{\Gamma\left(\frac{\alpha}{\rho}+1\right)}{\Gamma\left(\frac{\alpha}{\rho}+1+\lambda\right)}s^{\alpha+\rho\lambda} \tag{1.22}$$

and

$$\left({}^{\rho}I_{0-}^{\lambda}t^{\alpha}\right)(s) = (-\rho)^{-\lambda}\frac{\Gamma\left(\frac{\alpha}{\rho}+1\right)}{\Gamma\left(\frac{\alpha}{\rho}+1+\lambda\right)}s^{\alpha+\rho\lambda}. \tag{1.23}$$

Lemma 1.2 *Let $\rho > 0$, $\Re(\alpha) > 0$, $\Re(\lambda) > 0$, and $n = [\Re(\lambda)] + 1$. Then*

$$\left({}^{\rho}D_{0+}^{\lambda}t^{\alpha}\right)(s) = \rho^{\lambda}\frac{\Gamma\left(\frac{\alpha}{\rho}+1\right)}{\Gamma\left(\frac{\alpha}{\rho}+1-\lambda\right)}s^{\alpha-\rho\lambda} \tag{1.24}$$

and

$$\left({}^{\rho}D_{0-}^{\lambda}t^{\alpha}\right)(s) = (-\rho)^{\lambda}\frac{\Gamma\left(\frac{\alpha}{\rho}+1\right)}{\Gamma\left(\frac{\alpha}{\rho}+1-\lambda\right)}s^{\alpha-\rho\lambda}. \tag{1.25}$$

Numerous image formulae for a diversity of polynomials and functions subjected to a variety of fractional integrals and derivatives have been given (see,

e.g., [1], [2], [3], [4], [9], [18], [22], [23], [24], [25]). The purpose of this article isto establish image formulae for the product of incomplete H-functions and a general class of polynomials under the Katugampola fractional integral and derivative operators. Among many others, we also present some specific examples of our major results.

2 Katugampola fractional integral operators involving incomplete H-functions and general class of polynomials

In this part, we state the following theorems that establish the image formulae for product of the incomplete H-functions and the general class of polynomials under the left- and right-sided Katugampola fractional integral operators.

Theorem 2.1 *Let $\Re(\lambda) > 0$, a, $b \in \mathbb{R}$, ρ, α, $\beta \in \mathbb{R}^+$, $\mathsf{y} \geq 0$, and $s > 0$. Then*

$$\left({}^{\rho}I_{0+}^{\lambda} S_{\mathsf{n}}^{\mathsf{m}}[at^{\alpha}]\Gamma_{\mathsf{p,q}}^{\mathsf{m,n}}\left[bt^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{matrix}\right.\right]\right)(s) = \rho^{-\lambda}s^{\rho\lambda}\sum_{k=0}^{[\frac{\mathsf{n}}{\mathsf{m}}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^k$$
$$\times\Gamma_{\mathsf{p+1,q+1}}^{\mathsf{m,n+1}}\left[bs^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}(\frac{-\alpha k}{\rho},\frac{\beta}{\rho})\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(-\lambda-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{matrix}\right.\right] \tag{2.1}$$

and

$$\left({}^{\rho}I_{0+}^{\lambda} S_{\mathsf{n}}^{\mathsf{m}}[at^{\alpha}]\gamma_{\mathsf{p,q}}^{\mathsf{m,n}}\left[bt^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{q}}\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,w}\end{matrix}\right.\right]\right)(s) = \rho^{-\lambda}s^{\rho\lambda}\sum_{k=0}^{[\frac{\mathsf{n}}{\mathsf{m}}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^k$$
$$\times\gamma_{\mathsf{p+1,q+1}}^{\mathsf{m,n+1}}\left[bs^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}(\frac{-\alpha k}{\rho},\frac{\beta}{\rho})\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(-\lambda-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{matrix}\right.\right]. \tag{2.2}$$

Proof. Let Δ be the left-handed member of (2.1). Using (1.15), (1.10) and (1.8), and changing the order of integrals, which may be readily verified under the constraints, we have

$$\Delta = \sum_{k=0}^{[\mathsf{n}/\mathsf{m}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}\,a^k\int_{\mathfrak{C}}\mathbb{F}(\xi,\mathsf{y})b^{-\xi}\,\left({}^{\rho}I_{0+}^{\lambda}\left[t^{\alpha k-\beta\xi}\right]\right)(s)\,d\xi. \tag{2.3}$$

Employing (1.22) to evaluate the right-handed Katugampola fractional integral

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in (2.3), we obtain

$$\Delta = \rho^{-\lambda} s^{\rho\lambda} \sum_{k=0}^{[\mathsf{n}/\mathsf{m}]} \frac{(-\mathsf{n})_{\mathsf{m}k}}{k!} A_{\mathsf{n},k} (s^{\alpha} a)^k \frac{1}{2\pi i} \int_{\mathfrak{C}} \mathbb{F}(\xi, \mathsf{y}) (b s^{\beta})^{-\xi} \frac{\Gamma\left[1 + \frac{\alpha k}{\rho} - \frac{\beta}{\rho}\xi\right]}{\left[1 + \lambda + \frac{\alpha k}{\rho} - \frac{\beta}{\rho}\xi\right]} d\xi,$$

which, upon expressing the integral in terms of (1.8), yields the desired right-handed member of (2.1).

The proof of (2.2) would run in parallel with that of (2.1). We omit the specific. □

Theorem 2.2 *Let* $\Re(\lambda) > 0$, $a, b \in \mathbb{R}$, $\rho, \alpha, \beta \in \mathbb{R}^+$, $\mathsf{y} \geq 0$, *and* $s < 0$. *Then*

$$\left({}^{\rho}I_{0-}^{\lambda} S_{\mathsf{n}}^{\mathsf{m}}[a t^{\alpha}] \Gamma_{\mathsf{p},\mathsf{q}}^{\mathsf{m},\mathsf{n}} \left[b t^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}} \end{matrix} \right] \right)(s) = (-\rho)^{-\lambda} s^{\rho\lambda} \sum_{k=0}^{[\frac{\mathsf{n}}{\mathsf{m}}]} \frac{(-\mathsf{n})_{\mathsf{m}k}}{k!} A_{\mathsf{n},k} (a s^{\alpha})^k \times \Gamma_{\mathsf{p}+1,\mathsf{q}+1}^{\mathsf{m},\mathsf{n}+1} \left[b\, s^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} (-\frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}}, (-\lambda - \frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \end{matrix} \right] \tag{2.4}$$

and

$$\left({}^{\rho}I_{0-}^{\lambda} S_{\mathsf{n}}^{m}[a t^{\alpha}] \gamma_{\mathsf{p},\mathsf{q}}^{\mathsf{m},\mathsf{n}} \left[b t^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}} \end{matrix} \right] \right)(s) = (-\rho)^{-\lambda} s^{\rho\lambda} \sum_{k=0}^{[\frac{\mathsf{n}}{\mathsf{m}}]} \frac{(-\mathsf{n})_{\mathsf{m}k}}{k!} A_{\mathsf{n},k} (a s^{\alpha})^k \times \gamma_{\mathsf{p}+1,\mathsf{q}+1}^{\mathsf{m},\mathsf{n}+1} \left[b s^{\beta} \middle| \begin{matrix} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} (-\frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}}, (-\lambda - \frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \end{matrix} \right]. \tag{2.5}$$

Proof. The proof would proceed in the same manner as the proof of Theorem 2.1. We omit specifics. □

3 Katugampola fractional derivative operators with incomplete H-functions and general class of polynomials

The following two theorems provide the image formulae for product of the incomplete H-functions and the general class of polynomials under the left- and right-sided Katugampola fractional derivative operators. Since the proofs here would be identical to those used in Theorems 2.1 and 2.2, we omit the required proofs.

Theorem 3.1 *Let* $\Re(\lambda) > 0$, $a, b \in \mathbb{R}$, $\rho, \alpha, \beta \in \mathbb{R}^+$, $\mathsf{y} \geq 0$, *and* $s > 0$. *Then*

$$\left({}^{\rho}\mathcal{D}^{\lambda}_{0+} S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}]\Gamma^{\mathsf{m},\mathsf{n}}_{\mathsf{p},\mathsf{q}}\left[bt^{\beta}\,\middle|\,\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}\\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{matrix}\right]\right)(s)$$
$$= \rho^{\lambda}s^{\rho\lambda}\sum_{k=0}^{[\mathsf{n}/\mathsf{m}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^k\Gamma^{\mathsf{m},\mathsf{n}+1}_{\mathsf{p}+1,\mathsf{q}+1}\left[bs^{\beta}\,\middle|\,\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}},(-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(\lambda-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{matrix}\right] \tag{3.1}$$

and

$$\left({}^{\rho}\mathcal{D}^{\lambda}_{0+} S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}]\gamma^{\mathsf{m},\mathsf{n}}_{\mathsf{p},\mathsf{q}}\left[bt^{\beta}\,\middle|\,\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}\\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{matrix}\right]\right)(s)$$
$$= \rho^{\lambda}s^{\rho\lambda}\sum_{k=0}^{[\mathsf{n}/\mathsf{m}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^k\gamma^{\mathsf{m},\mathsf{n}+1}_{\mathsf{p}+1,\mathsf{q}+1}\left[bs^{\beta}\,\middle|\,\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}},(-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(\lambda-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{matrix}\right]. \tag{3.2}$$

Theorem 3.2 *Let* $\Re(\lambda) > 0$, $a, b \in \mathbb{R}$, $\rho, \alpha, \beta \in \mathbb{R}^+$, $\mathsf{y} \geq 0$, *and* $s < 0$. *Then*

$$\left({}^{\rho}\mathcal{D}^{\lambda}_{0-} S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}]\Gamma^{\mathsf{m},\mathsf{n}}_{\mathsf{p},\mathsf{q}}\left[bt^{\beta}\,\middle|\,\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}\\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{matrix}\right]\right)(s)$$
$$= (-\rho)^{\lambda}s^{\rho\lambda}\sum_{k=0}^{[\mathsf{n}/\mathsf{m}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^k\Gamma^{\mathsf{m},\mathsf{n}+1}_{\mathsf{p}+1,\mathsf{q}+1}\left[bs^{\beta}\,\middle|\,\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}},(-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(\lambda-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{matrix}\right] \tag{3.3}$$

and

$$\left({}^{\rho}\mathcal{D}^{\lambda}_{0-} S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}]\gamma^{\mathsf{m},\mathsf{n}}_{\mathsf{p},\mathsf{q}}\left[bt^{\beta}\,\middle|\,\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}\\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{matrix}\right]\right)(s)$$
$$= (-\rho)^{\lambda}s^{\rho\lambda}\sum_{k=0}^{[\mathsf{n}/\mathsf{m}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^k\gamma^{\mathsf{m},\mathsf{n}+1}_{\mathsf{p}+1,\mathsf{q}+1}\left[bs^{\beta}\,\middle|\,\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}},(-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(\lambda-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{matrix}\right]. \tag{3.4}$$

4 Particular cases and remarks

Due to the generality of both incomplete H-functions and the general class polynomials, the main identities established in the preceding sections may result

in a variety of simpler formulae as special instances. For example, the case $\mathsf{y} = 0$ of (1.8) reduces to the Fox's H-function (see, e.g., [20, p. 10]; see also [11], [16]):

$$\Gamma^{\mathsf{m},\mathsf{n}}_{\mathsf{p},\mathsf{q}}\left[z \,\middle|\, \begin{matrix} (\mathsf{e}_1,\mathsf{E}_1),(\mathsf{e}_i,\mathsf{E}_i)_{2,\mathsf{p}} \\ \\ (\mathsf{f}_i,\mathsf{F}_i)_{1,\mathsf{q}} \end{matrix} \right] = H^{\mathsf{m},\mathsf{n}}_{\mathsf{p},\mathsf{q}}\left[z \,\middle|\, \begin{matrix} (\mathsf{e}_1,\mathsf{E}_1),(\mathsf{e}_i,\mathsf{E}_i)_{2,\mathsf{p}} \\ \\ (\mathsf{f}_i,\mathsf{F}_i)_{1,\mathsf{q}} \end{matrix} \right]$$
$$= H^{\mathsf{m},\mathsf{n}}_{\mathsf{p},\mathsf{q}}\left[z \,\middle|\, \begin{matrix} (\mathsf{e}_1,\mathsf{E}_1),(\mathsf{e}_2,\mathsf{E}_2),\cdots,(\mathsf{e}_\mathsf{p},\mathsf{E}_\mathsf{p}) \\ \\ (\mathsf{f}_1,\mathsf{F}_1),(\mathsf{f}_2,\mathsf{F}_2),\cdots,(\mathsf{f}_\mathsf{q},\mathsf{F}_\mathsf{q}) \end{matrix} \right]. \tag{4.1}$$

For another example, putting $\mathsf{m} = 1$, $\mathsf{n} = \mathsf{p}$, q being replaced by $\mathsf{q}+1$ and taking appropriate parameters, the functions (1.6) and (1.8) reduce, respectively, to the incomplete Fox-Wright Ψ-functions ${}_\mathsf{p}\Psi^{(\gamma)}_\mathsf{q}$ and ${}_\mathsf{p}\Psi^{(\Gamma)}_\mathsf{q}$ (see [21, Eqs. (6.3) and (6.4)]; see also [2, Eqs. (14) and (15)]):

$$\gamma^{1,\mathsf{p}}_{\mathsf{p},\mathsf{q}+1}\left[-z \,\middle|\, \begin{matrix} (1-\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(1-\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}} \\ \\ (0,1),(1-\mathsf{b}_j,\mathsf{B}_j)_{1,\mathsf{q}} \end{matrix} \right] = {}_\mathsf{p}\Psi^{(\gamma)}_\mathsf{q}\left[\begin{matrix} (\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}; \\ \\ (\mathsf{b}_j,\mathsf{B}_j)_{1,\mathsf{q}}; \end{matrix} \; z \right] \tag{4.2}$$

and

$$\Gamma^{1,\mathsf{p}}_{\mathsf{p},\mathsf{q}+1}\left[-z \,\middle|\, \begin{matrix} (1-\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(1-\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}} \\ \\ (0,1),(1-\mathsf{b}_j,\mathsf{B}_j)_{1,\mathsf{q}} \end{matrix} \right] = {}_\mathsf{p}\Psi^{(\Gamma)}_\mathsf{q}\left[\begin{matrix} (\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}; \\ \\ (\mathsf{b}_j,\mathsf{B}_j)_{1,\mathsf{q}}; \end{matrix} \; z \right]. \tag{4.3}$$

The following corollaries cover some of them.

Corollary 4.1 *Let* $\Re(\lambda) > 0$, a, $b \in \mathbb{R}$, *and* ρ, α, $\beta \in \mathbb{R}^+$. *Then*

$$\left({}^\rho I^\lambda_{0+} S^\mathsf{m}_\mathsf{n}[at^\alpha] H^{\mathsf{m},\mathsf{n}}_{\mathsf{p},\mathsf{q}}\left[bt^\beta \,\middle|\, \begin{matrix} (\mathsf{e}_j,\mathsf{E}_j)_{1,\mathsf{p}} \\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}} \end{matrix} \right] \right)(s) = \rho^{-\lambda} s^{\rho\lambda} \sum_{k=0}^{[\mathsf{n}/\mathsf{m}]} \frac{(-\mathsf{n})_{\mathsf{m}k}}{k!} A_{\mathsf{n},k} (as^\alpha)^k$$
$$\times H^{\mathsf{m},\mathsf{n}+1}_{\mathsf{p}+1,\mathsf{q}+1}\left[bs^\beta \,\middle|\, \begin{matrix} (\mathsf{e}_j,\mathsf{E}_j)_{1,\mathsf{p}}(-\frac{\alpha k}{\rho},\frac{\beta}{\rho}) \\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(-\lambda-\frac{\alpha k}{\rho},\frac{\beta}{\rho}) \end{matrix} \right] \tag{4.4}$$

$$(s > 0)$$

and

$$\left({}^\rho I^\lambda_{0-} S^\mathsf{m}_\mathsf{n}[at^\alpha] H^{\mathsf{m},\mathsf{n}}_{\mathsf{p},\mathsf{q}}\left[bt^\beta \,\middle|\, \begin{matrix} (\mathsf{e}_j,\mathsf{E}_j)_{1,\mathsf{p}} \\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}} \end{matrix} \right] \right)(s) = (-\rho)^{-\lambda} s^{\rho\lambda} \sum_{k=0}^{[\mathsf{n}/\mathsf{m}]} \frac{(-\mathsf{n})_{\mathsf{m}k}}{k!} A_{\mathsf{n},k} (as^\alpha)^k$$
$$\times H^{\mathsf{m},\mathsf{n}+1}_{\mathsf{p}+1,\mathsf{q}+1}\left[bs^\beta \,\middle|\, \begin{matrix} (\mathsf{e}_j,\mathsf{E}_j)_{1,\mathsf{p}}(\frac{-\alpha k}{\rho},\frac{\beta}{\rho}) \\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(-\lambda-\frac{\alpha k}{\rho},\frac{\beta}{\rho}) \end{matrix} \right] \tag{4.5}$$

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$$(s<0).$$

Proof. Taking $\mathsf{y}=0$ in (2.1) and (2.4), we get the required results. □

Corollary 4.2 *Let $\Re(\lambda)>0$, a, $b\in\mathbb{R}$, and ρ, α, $\beta\in\mathbb{R}^+$. Then*

$$\left({}^{\rho}\mathcal{D}^{\lambda}_{0+}S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}]H^{\mathsf{m,n}}_{\mathsf{p,q}}\left[bt^{\beta}\middle|\begin{matrix}(\mathsf{e}_j,\mathsf{E}_j)_{1,\mathsf{p}}\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{matrix}\right]\right)(s)=\rho^{\lambda}s^{\rho\lambda}\sum_{k=0}^{[\frac{\mathsf{n}}{\mathsf{m}}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^k \times H^{\mathsf{m,n+1}}_{\mathsf{p+1,q+1}}\left[bs^{\beta}\middle|\begin{matrix}(\mathsf{e}_j,\mathsf{E}_j)_{1,\mathsf{p}},(-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(\lambda-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{matrix}\right] \tag{4.6}$$

$$(s>0)$$

and

$$\left({}^{\rho}\mathcal{D}^{\lambda}_{0-}S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}]H^{\mathsf{m,n}}_{\mathsf{p,q}}\left[bt^{\beta}\middle|\begin{matrix}(\mathsf{e}_j,\mathsf{E}_j)_{1,\mathsf{p}}\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{matrix}\right]\right)(s)=\rho^{\lambda}s^{\rho\lambda}\sum_{k=0}^{[\mathsf{n}/\mathsf{m}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^k \times H^{\mathsf{m,n+1}}_{\mathsf{p+1,q+1}}\left[bs^{\beta}\middle|\begin{matrix}(\mathsf{e}_j,\mathsf{E}_j)_{1,\mathsf{p}},(-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(\lambda-\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{matrix}\right] \tag{4.7}$$

$$(s<0).$$

Proof. Taking $\mathsf{y}=0$ in (3.1) and (3.3), we get the required results. □

Corollary 4.3 *Let $\Re(\lambda)>0$, a, $b\in\mathbb{R}$, ρ, α, $\beta\in\mathbb{R}^+$, $\mathsf{y}\geq 0$, and $s>0$. Then*

$$\left({}^{\rho}I^{\lambda}_{0+}S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}]_{\mathsf{p}}\Psi^{(\Gamma)}_{\mathsf{q}}\left[bt^{\beta}\middle|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{matrix}\right]\right)(s)=\rho^{-\lambda}s^{\rho\lambda}\sum_{k=0}^{[\mathsf{n}/\mathsf{m}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^k \times {}_{\mathsf{p+1}}\Psi^{(\Gamma)}_{\mathsf{q+1}}\left[bs^{\beta}\middle|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}(1+\frac{\alpha k}{\rho},\frac{\beta}{\rho})\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(1+\lambda+\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{matrix}\right] \tag{4.8}$$

and

$$\left({}^{\rho}I^{\lambda}_{0+}S^{\mathsf{m}}_{\mathsf{n}}[at^{\alpha}]_{\mathsf{p}}\Psi^{(\gamma)}_{\mathsf{q}}\left[bt^{\beta}\middle|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{matrix}\right]\right)(s)=\rho^{-\lambda}s^{\rho\lambda}\sum_{k=0}^{[\mathsf{n}/\mathsf{m}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^k \times {}_{\mathsf{p+1}}\Psi^{(\gamma)}_{\mathsf{q+1}}\left[bs^{\beta}\middle|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}(1+\frac{\alpha k}{\rho},\frac{\beta}{\rho})\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(1+\lambda+\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{matrix}\right]. \tag{4.9}$$

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Proof. Using (4.2) and (4.3) in (2.1) and (2.2) gives the required identities. □

Corollary 4.4 *Let $\Re(\lambda) > 0$, a, $b \in \mathbb{R}$, ρ, α, $\beta \in \mathbb{R}^+$, $\mathsf{y} \geq 0$, and $s < 0$. Then*

$$\left({}^{\rho}I_{0-}^{\lambda} S_{\mathsf{n}}^{\mathsf{m}}[at^{\alpha}]_{\mathsf{p}}\Psi_{\mathsf{q}}^{(\Gamma)}\left[bt^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{matrix}\right.\right]\right)(s)$$
$$= (-\rho)^{-\lambda}s^{\rho\lambda}\sum_{k=0}^{[\mathsf{n}/\mathsf{m}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^{k}{}_{\mathsf{p}+1}\Psi_{\mathsf{q}+1}^{(\Gamma)}\left[bs^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}(1+\frac{\alpha k}{\rho},\frac{\beta}{\rho})\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(1+\lambda+\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{matrix}\right.\right] \tag{4.10}$$

and

$$\left({}^{\rho}I_{0-}^{\lambda} S_{\mathsf{n}}^{\mathsf{m}}[at^{\alpha}]_{\mathsf{p}}\Psi_{\mathsf{q}}^{(\gamma)}\left[bt^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{matrix}\right.\right]\right)(s)$$
$$= (-\rho)^{-\lambda}s^{\rho\lambda}\sum_{k=0}^{[\frac{\mathsf{n}}{\mathsf{m}}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^{k}{}_{\mathsf{p}+1}\Psi_{\mathsf{q}+1}^{(\gamma)}\left[bs^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}(1+\frac{\alpha k}{\rho},\frac{\beta}{\rho})\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(1+\lambda+\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{matrix}\right.\right] \tag{4.11}$$

Proof. Employing (4.2) and (4.3) in (2.4) and (2.5) provides the desired identities. □

Corollary 4.5 *Let $\Re(\lambda) > 0$, a, $b \in \mathbb{R}$, ρ, α, $\beta \in \mathbb{R}^+$, $\mathsf{y} \geq 0$, and $s > 0$. Then*

$$\left({}^{\rho}\mathcal{D}_{0+}^{\lambda} S_{\mathsf{n}}^{\mathsf{m}}[at^{\alpha}]_{\mathsf{p}}\Psi_{\mathsf{q}}^{(\Gamma)}\left[bt^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{matrix}\right.\right]\right)(s)$$
$$= \rho^{\lambda}s^{\rho\lambda}\sum_{k=0}^{[\mathsf{n}/\mathsf{m}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^{k}{}_{\mathsf{p}+1}\Psi_{\mathsf{q}+1}^{(\Gamma)}\left[bs^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}},(1+\frac{\alpha k}{\rho},\frac{\beta}{\rho})\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(1-\lambda+\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{matrix}\right.\right] \tag{4.12}$$

and

$$\left({}^{\rho}\mathcal{D}_{0+}^{\lambda} S_{\mathsf{n}}^{\mathsf{m}}[at^{\alpha}]_{\mathsf{p}}\Psi_{\mathsf{q}}^{(\gamma)}\left[bt^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}}\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}}\end{matrix}\right.\right]\right)(s)$$
$$= \rho^{\lambda}s^{\rho\lambda}\sum_{k=0}^{[\frac{\mathsf{n}}{\mathsf{m}}]}\frac{(-\mathsf{n})_{\mathsf{m}k}}{k!}A_{\mathsf{n},k}(as^{\alpha})^{k}{}_{\mathsf{p}+1}\Psi_{\mathsf{q}+1}^{(\Gamma)}\left[bs^{\beta}\left|\begin{matrix}(\mathsf{e}_1,\mathsf{E}_1,\mathsf{y}),(\mathsf{e}_j,\mathsf{E}_j)_{2,\mathsf{p}},(1+\frac{\alpha k}{\rho},\frac{\beta}{\rho})\\ \\ (\mathsf{f}_j,\mathsf{F}_j)_{1,\mathsf{q}},(1-\lambda+\frac{\alpha k}{\rho},\frac{\beta}{\rho})\end{matrix}\right.\right]. \tag{4.13}$$

Proof. Applying (4.2) and (4.3) to (3.1) and (3.2) offers the desired results. □

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Corollary 4.6 *Let $\Re(\lambda) > 0$, $\delta \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}$, $a, b \in \mathbb{R}$, $\rho, \alpha, \beta \in \mathbb{R}^+$, $\mathsf{y} \geq 0$, and $s > 0$. Then*

$$\left({}^{\rho}I_{0+}^{\lambda} L_{\mathsf{n}}^{(\delta)}(at^{\alpha}) \Gamma_{\mathsf{p},\mathsf{q}}^{1,\mathsf{n}} \left[bt^{\beta} \middle| \begin{array}{c} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}} \end{array} \right] \right)(s) = \rho^{-\lambda} s^{\rho\lambda} \sum_{k=0}^{[\mathsf{n}]} \frac{(-\mathsf{n})_k}{k!\, \mathsf{n}!} \frac{(1+\delta)_{\mathsf{n}}}{(1+\delta)_k} (as^{\alpha})^k \times \Gamma_{\mathsf{p}+1,\mathsf{q}+1}^{1,\mathsf{n}+1} \left[bs^{\beta} \middle| \begin{array}{c} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} (-\frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}}, (-\lambda - \frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \end{array} \right] \tag{4.14}$$

and

$$\left({}^{\rho}I_{0+}^{\lambda} L_{\mathsf{n}}^{(\delta)}(at^{\alpha}) \gamma_{\mathsf{p},\mathsf{q}}^{1,\mathsf{n}} \left[bt^{\beta} \middle| \begin{array}{c} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{q}} \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,w} \end{array} \right] \right)(s) = \rho^{-\lambda} s^{\rho\lambda} \sum_{k=0}^{[\mathsf{n}]} \frac{(-\mathsf{n})_k}{k!\, \mathsf{n}!} \frac{(1+\delta)_{\mathsf{n}}}{(1+\delta)_k} (as^{\alpha})^k \times \gamma_{\mathsf{p}+1,\mathsf{q}+1}^{1,\mathsf{n}+1} \left[bs^{\beta} \middle| \begin{array}{c} (\mathsf{e}_1, \mathsf{E}_1, \mathsf{y}), (\mathsf{e}_j, \mathsf{E}_j)_{2,\mathsf{p}} (-\frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \\ (\mathsf{f}_j, \mathsf{F}_j)_{1,\mathsf{q}}, (-\lambda - \frac{\alpha k}{\rho}, \frac{\beta}{\rho}) \end{array} \right], \tag{4.15}$$

where $L_{\mathsf{n}}^{(\delta)}(x)$ are Laguerre polynomials.

Proof. Setting $\mathsf{m} = 1$ and choosing $A_{\mathsf{n},k} = (1+\delta)_{\mathsf{n}} / \{(1+\delta)_k\, \mathsf{n}!\}$ in the results in Theorem 2.1, with the aid of Laguerre polynomials $L_{\mathsf{n}}^{(\delta)}(x)$ (see, e.g., [15, p. 201, Eq. (3)]), we obtain the desired identities here. □

Likewise, as with Corollary 4.6, substituting $\mathsf{m} = 1$ and selecting $A_{\mathsf{n},k} = (1+\delta)_{\mathsf{n}} / \{(1+\delta)_k\, \mathsf{n}!\}$ in the identities in Theorems 2.2–3.2 and Corollaries 4.1–4.5 results in the corresponding formulae involving the Laguerre polynomials.

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New Fixed Points Outcomes for Fractal Creation by Applying Different Fixed Point Technique

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Abstract

Fractal like Julia set is regarded as one of the striking and significant mathematical fractals in the field of science and technology. There are different numerical iterative techniques which generate these fractals and in fact these numerical iterative techniques are the strength of fractal geometry. In recent past, Julia sets have been studied through numerical techniques like Picard, Mann, and Ishikawa etc. which are the examples of one-step, two-step, and three-step iterative techniques respectively. In this article, we have concentrated our research work on the computation as well as the different features of cubic Julia sets for the complex polynomial $P_{m,n}(z) = z^3 + mz + n$. This generation process has been carried-over through a new numerical four-step iterative technique. We have generated new and ever seen cubic Julia sets for the above complex polynomial. The cubic Julia sets generated through above polynomial have important mathematical properties. It is also fascinating that some of the generated cubic Julia sets are analogous to fractal shaped antennas, butterfly and some categories of ants. Some of the generated cubic Julia sets can also be categorized as wall-decorated pictures.

Key words: Cubic Julia sets; Four-Step Iterative Technique; Escape Criterion; Complex Cubic Equation

Mathematics Subject Classification(2010): 37C25; 28A80; 54E35; 54E50

1

1 Introduction

Fractals have many applications in various fields of science and technology. The production of cell phone has become possible only due to fractal shaped antennas. This was not possible before the introduction of this notion. The Chaos theory is what fractals are all about, and that is at the core of most Cosmological Models which describe our entire Universe on both the scales of very small and large. Fractal geometry is very practical for all sorts of things from biology to physics to cosmology to even the stock market. For instance veins and arteries form fractal trees, some mineral deposits are distributed under the fractal laws through the soil, and the stock market, perhaps not quite a natural phenomenon, behaves according to fractal laws.

If we look back into the history of fractal geometry, we find that the interest in fractal geometry with respect to Julia sets began in the 19th century. Named after

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the esteemed French mathematician Gaston Julia, the Julia set expresses a complex and rich set of dynamics, giving rise to a large range of breathtaking visuals. Gaston was first who invented Julia sets and examined their features [27] $(p-122)$. In 1918, a masterpiece of him on Julia sets was published which was the major event in the history of fractal geometry. In this masterpiece, Gaston Julia first proposed the latest idea on a Julia set and that made him famous in the world of mathematics. Julia sets live in complex plane and this is the place where all chaotic behavior of complex plane occurs [9] $(p-221)$. When all the computational experiments were far ahead of its time, Herald Cramer gave the first approximate graphical representation to Julia sets. Mathematical items like Julia sets were recognized when computer graphics became user friendly [28].

What makes Julia sets interesting to analysis is that, despite being borne out of apparently easy iterative techniques, they can be very tangled and often fractal in nature. Due to fascination of computer experiments and incredulity of their graphical representation, Mandelbrot and Julia asets remain a topic of modern research with much interest being in evoking the tangled structure of Julia sets and in calculating their properties such as 'fractal dimension' [14]. For detailed investigation on Mandelbrot and Julia sets, one can follow the related work and complex dynamics system in references like [4, 5, 6, 8, 9, 12, 16, 18, 25, 26, 39].These sets (Julia sets) have been computed and examined for cubic [2, 3, 7, 9, 12, 13], quadratic [9, 19, 27], and also for higher degree polynomials using Picard iteration,which is an example of one-step numerical procedure.

Julia sets may be generated for any order complex polynomial. In this research Paper, we have emphasized our research work on cubic Julia sets with reference to complex polynomial of order 3. In 1988, Branner and Hubbard, by a series of papers, paid concerned with the way in which the space of polynomial of degree $d > 3$ is decomposed when the polynomial are classified according to their active behavior using iterative procedure. However their results are satisfied only for $d = 3$. They also dispense with the good framework of the locus connectedness for cubic polynomials [2]. In 1992, their second paper of this series was also dedicated to the account of dynamical system complex cubic polynomial. They also raised the query when the Julia set of a cubic polynomial became a Cantor set [3].

In 1998, Liaw found the regularities for the parameter space of the cubic polynomials in the two dimensional projections. The projection of the parameter points that have non-totally disconnected Julia sets can be observe as a combination of Mandelbrot-like sets [20]. In the same year Cheng and Liaw established asymptotic similarity between the parameter spaces (the Mandelbrot set) and the dynamic space (the Julia sets) of cubic mappings. The dynamic spaces and the parametric spaces of cubic polynomials consist many small copies of the Julia sets and Mandelbrot sets respectively of the standard quadratic mapping [7]. In 2001, Liaw studied the orientations, sizes, and positions of above small copies to understand the formation of the cubic polynomials [21].

In 1999, Yan, Liu, and Zhu worked on a general complex cubic iteration and produced results on the range of Mandelbrot and Julia sets generated from above cubic iteration[41]. These results are helpful in plotting the above generated sets. In 2003, Tomova produced results about the limits of cubic Julia and Mandelbrot sets for complex polynomial $z^3 + pz + q$ and also for Julia and Mandelbrot sets of higher orders polynomial of the form $z^n + c$, [40].

In 2006, Fu *et.al.* generated higher order Julia set and Mandelbrot sets by applying fast computing algorithm [15]. Mamta *et.al.* studied and computed superior Julia sets for nth degree complex polynomial [32, 33], for cubic [6] and for quadratic complex

polynomials [17, 30, 31] using superior iterates (a two-step feedback procedure). Superior Julia sets have also been analysed under the effect of noises [34, 35]. In 2018, Narayan*et.al.* computed and analysed new collection of antifractals for the complex polynomial $\bar{z}^n + c$ in the GK-orbit [23].

In our previous article, we have computed and examined new collection of fractals using faster iteration with s-convexity [24].Recently, Rahman, Nisar and Golamankaneh, also studied the generalized Riemann–Liouville fractional integral for the functions with fractal support. They explored reverse Minkowski's inequalities and certain other related inequalities by employing generalized Riemann–Liouville fractional integral for the functions with fractal support [29].

Fractals have also vital role in various domains ranging from mathematical to engineering applications, nano technology to bio medical domain. However, for a deep understanding of fractals and to review the relevant areas, one must go through some recent implementations of non-linear concepts in various areas of machine learning, microprocessors and computer science related to data analysis [10, 37]. For more applications of numerical computing techniques one may see [11, 38].

In this research paper, we have explored new and ever seen cubic fractal sets for the complex cubic polynomial $z \to z^3 + mz + n$, using new different numerical iterative technique (a four-step feedback system) with improved escape criterion. We have also analysed the characteristics of the above generated cubic Julia sets.

2 Material and Method

In past years, researchers focused mainly on three types of feedback procedures like Picard iterates, a one-step feedback system [27], Mann iterates, a two-step feedback system [17, 36], Ishikawa iterates, a three-step feedback system [6] etc. In 2014, Mujahid *et. al.* introduced a new four-step iterative procedure and proved that it is faster than all of Mann, Picard and Agarwal *et. al.* processes. They supported analytic proof by a numerical example and mentioned that this process is independent of all above processes. They also demonstrated some strong and weak convergence results for two non-expansive functions [1].

In this research paper, we have computed Julia sets for the cubic complex polynomial using this faster four-step feedback procedure with improved escape criterion which provides the different results than the previous results.

Definition 2.1. Let S be a non-empty set and p be a self-map from S to S. For a point s_0 in S, the **Picard orbit** (generally called orbit of p) is the set of all iterates of a point s_0, that is: $P(p,s_0) = \{s_n : s_n = p(s_{n-1}),\ n = 1,2,\cdots\}$, where $P(p,s_0)$ of p at the initial point s_0 is the sequence $(p^n s_0)$

Definition 2.2. The filled in Julia set of the function $F(z) = z^n + c$ is defined as

$$K(F) = \{z \in C : F^k(z) \text{ does not tend to } \infty\}$$

where $F^k(z)$ is the k^{th} iterate of function F, *K(F)* denotes the filled in Julia set and C is the complex space. The boundary of *K(F)* is known as Julia set of the function F. Julia set is the set of those points whose orbits are bounded under $F_c(z) = z^n + c$.

Definition 2.3. **(New Iterative Procedure-NIP)** Let X be a non-empty set such that $T : X \to X$ and $\{x_n\}$ be a sequence of iterates of initial point $x_0 \in X$ such that

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$$\{x_{n+1} : x_{n+1} = (1-u_n)Ty_n + u_nTz_n \ ;$$
$$y_n = (1-v_n)Tx_n + v_nTz_n \ ;$$
$$z_n = (1-w_n)x_n + w_nTx_n; \ n = 0,1,2,\cdots,\},$$

where $u_n, v_n, w_n \in [1,0]$ and $\{u_n\},\{v_n\},\{w_n\}$ are sequences of positive numbers. For the sake of simplicity, we take $u_n = u, v_n = v, w_n = w$.

2.1 Escape Criterion for Cubic Complex Polynomial

In computation of fractals like Julia and Mandelbrot sets, there is always need to establish a criterion called 'Escape criterion' which enacts a chief role in the graphical representation of Mandelbrot as well as Julia sets and these criteria are different for different order complex polynomials.
Here, in this part, we demonstrate the new escape criterion using NIP for the complex cubic polynomials:

$$P_{m,n}(z) = z^3 + mz + n$$

where m and n are complex numbers.
The following theorem and results provide the escape criterion using NIP for the above mentioned complex polynomial.

Theorem 2.1. *Consider that* $|z| > |n| > (|m| + 2/|u|)^{1/2}$, $|z| > |n| > (|m| + 2/|v|)^{1/2}$, *and* $|z| > |n| > (|m| + 2/|w|)^{1/2}$, *where* $0 < u < 1$, $0 < v < 1$, $0 < w < 1$, *and c is in the complex plane.*
Define

$$z_1 = (1-u)P_{m,n}(z) + uP_{m,n}(z) \ ,$$
$$z_2 = (1-u)P_{m,n}(z_1) + uP_{m,n}(z_1) \ ,$$
$$\vdots$$
$$z_n = (1-u)P_{m,n}(z_{n-1}) + uP_{m,n}(z_{n-1}) \ , n = 2,3,4,\cdots$$

where $P_{m,n}(z)$ *is the fuction of z.*
Then $|z| \to \infty$ *as* $n \to \infty$

Proof. Consider

$$\begin{aligned}
|P'_{m,n}(z)| &= |(1-w)z + wP''_{m,n}(z)| \ \ where P''_{m,n}(z) = z^3 + mz + n \\
&= |(1-w)z + w(z^3 + mz + n)| \\
&= |z - wz + wz^3 + mwz + wn)| \\
&\geq |wz^3 + mwz + z - wz| - |wn| \\
&\geq |z|(|wz^2 + mw + 1 - w|) - w|z|, && (\because |z| \geq n) \\
&\geq |z|(|wz^2 + mw| - |1 - w|) - w|z| \\
&= |z|(w|z^2 + m| - 1)
\end{aligned}$$

i.e.

$$|P'_{m,n}(z)| \geq |z|(w|z^2 + m| - 1) \tag{2.1}$$

Also

$$
\begin{aligned}
|P_{m,n}(z)| &= |(1-v)P''_{m,n}(z) + vP'_{m,n}(z)| \\
&\geq |(1-v)(z^3+mz+n+v|z|(w|z^2+m|-1|)) \qquad [By\ using\ Eq.(2.1)] \\
&= |(1-v)z^3+m(1-v)z+n(1-v)+vw|z|(|z^2+m|-v|z||) \\
&\geq |(1-v)z^3+m(1-v)z|-(1-v)|z|+vw|z|(|z^2+m|-v|z|) \\
&= |z|[(1-v)|z^2+m|-1]+vw|z||z^2+m| \\
&\geq |z|[(vw-v+1)(|z^2|-|m|-1)]
\end{aligned}
$$

Since,

$$z_n = (1-u)P_{m,n}(z_{n-1}) + uP'_{m,n}(z_{n-1})$$

We have

$$
\begin{aligned}
|z_1| &= |(1-u)P_{m,n}(z) + uP'_{m,n}(z)| \\
&\geq |(1-u)[|z|(vw-v+1)|z^2+m|-1]+u[|z|(w|z^2+m|-1)] \\
&\geq |z||(uv+vw-uvw-u-v+1)|z^2+m|-|(1-u)|z|| \\
&\quad +uw|z||z^2+m|-u|z| \\
&= |z|[(uv+vw+uw-uvw-u-v+1)|z^2+m|-1] \\
&\geq |z|[(uv+vw+uw-uvw-u-v+1)(|z^2|-|m|)-1] \\
&= |z|R[|z^2|-(|m|+1/R)]
\end{aligned}
$$

where

$$R = uv+vw+uw-uvw-u-v+1$$

Since, $|z| > |n| > (|m|+2/|u|)^{1/2}$, $|z| > |n| > (|m|+2/|v|)^{1/2}$, and $|z| > |n| > (|m|+2/|w|)^{1/2}$, so that we have $|z| > (|m|+2/R)^{1/2}$, this mean $|z^2|-(|m|+1/R) > (1/R)$ so that $R(|z^2|-(|m|+1/R)) > 1$. Therefore there exists a $\lambda > 0$ such that

$$|z_1| > (1+\lambda)|z|$$

Repeating the same argument, we get

$$|z_n| > (1+\lambda)^n|z|$$

Thus, orbit of z tends to infinity as $n \to \infty$. This completes the proof. □

Corollary 2.1.1. *Consider the complex cubic polynomial $P_{m,n}(z) = z^3+mz+n$ where m, n are complex numbers and assume $|z| > max[\ |n|, (|m|+2/|u|)^{1/2}, (|m|+2/|v|)^{1/2}, (|m|+2/|w|)^{1/2}]$ then $|z_n| > (1+\lambda)^n|z|$ and $|z_n| \to \infty$ as $n \to \infty$. This provides the 'escape criterion' for the above complex cubic polynomial.*

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Corollary 2.1.2. *Assume*
$|z_k| > max[\ |n|, (|m|+2/|u|)^{1/2}, (|m|+2/|v|)^{1/2}, (|m|+2/|w|)^{1/2}]$ *for some* $k \geq 0$ *then* $|z_k| > (1+\lambda)^n |z_{k-1}|$ *and* $|z_n| \to \infty$ *as* $n \to \infty$.
This result provides an algorithm to generate cubic Julia sets for the above mentioned complex polynomial.

3 Result And Discussion

3.1 Fractals as Cubic Julia sets

Cubic Julia sets, applying new different fixed point technique, are generated through complex polynomial $P_{m,n}(z) = z^3 + mz + n$, using cubic escape criterion with the software Mathematica 10.0, See Figures 1-20.

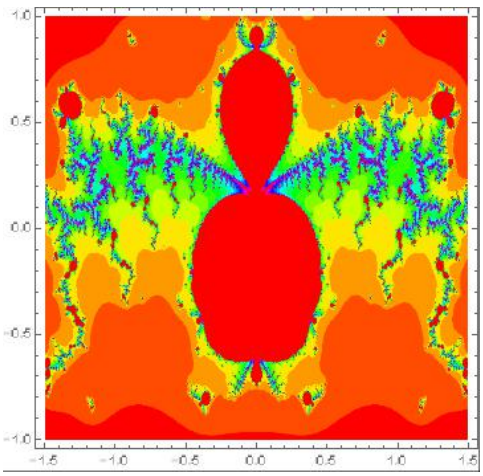

Figure 1: Ants Cubic Julia set for u= v=.48, w=.28, m=-3.2, n=.56I

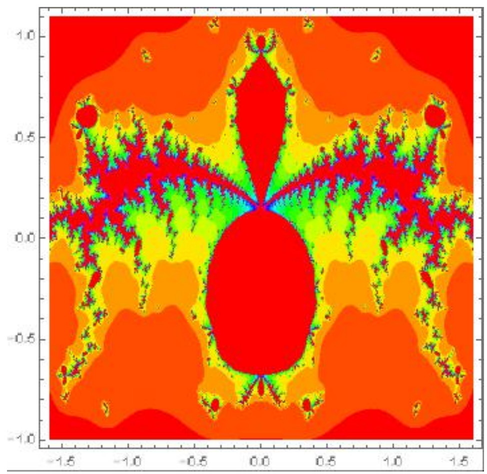

Figure 2: Ants Cubic Julia set for u= v=.477, w=.283, m=-3.13, n=.59I

3.2 Graphical Execution of Cubic Julia Sets Employed NIP

For the graphical execution of cubic Julia sets we iterate complex cubic polynomial $P_{m,n}(z) = z^3 + mz + n$, and define the prisoner set using escape criterion under the above new different iterative procedure. We have also generated some interested and ever seemed cubic Julia sets.

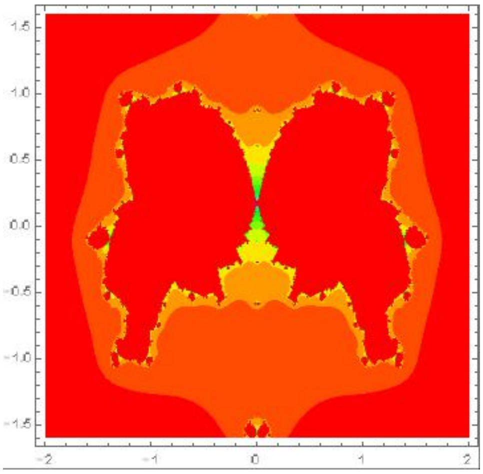

Figure 3: Cubic Julia set for u= v=.42, w=.05, m=-1.6, n=.45I

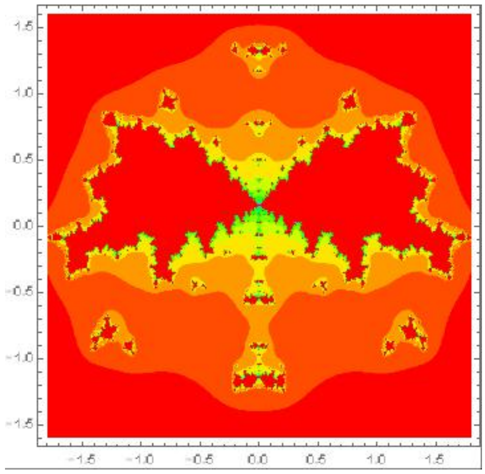

Figure 4: Cubic Julia set for u= v=.45, w=.09, m=-2.3, n=.52I

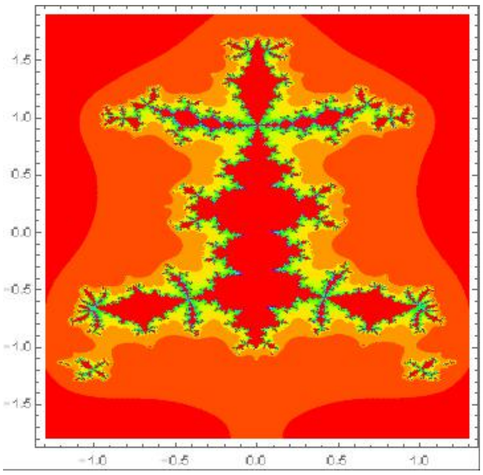

Figure 5: Cubic Julia set for u= v=.44 w=.07 m=.6-.01I n=1.2I

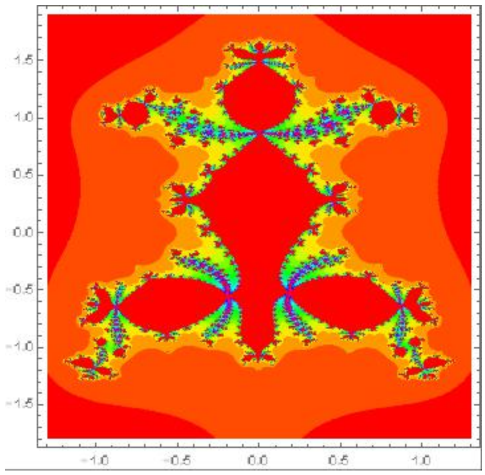

Figure 6: Cubic Julia set for u= v=.44 w=.03 m=.62-.01I n=1I

In this paper we have used two sets of parameters, $u, v, w$and m, n, and by changing parametric cost of these set of pairs, we have also noticed many important observations as follow:

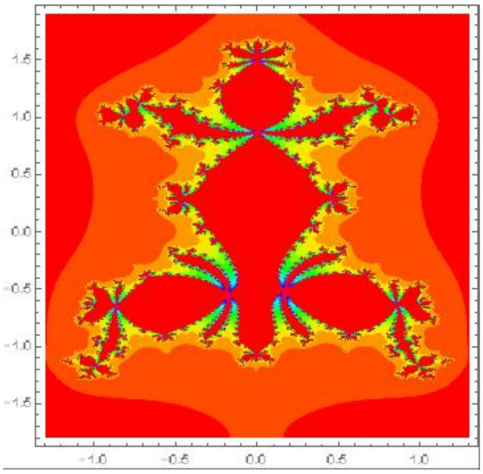

Figure 7: Cubic Julia set for u= v=.44 w=.07 m=.62-.01I n=1I

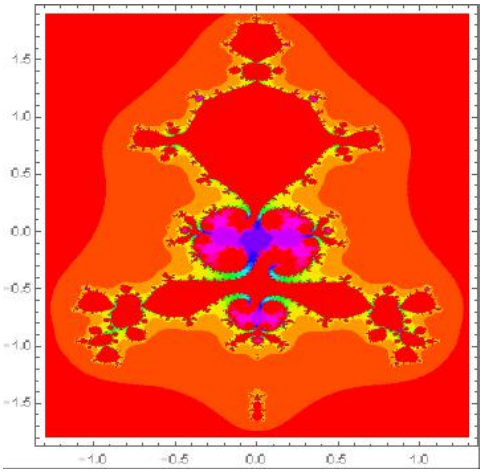

Figure 8: Cubic Julia set for u= v=.57 w=.05 m=.6+.01I n=1.2I

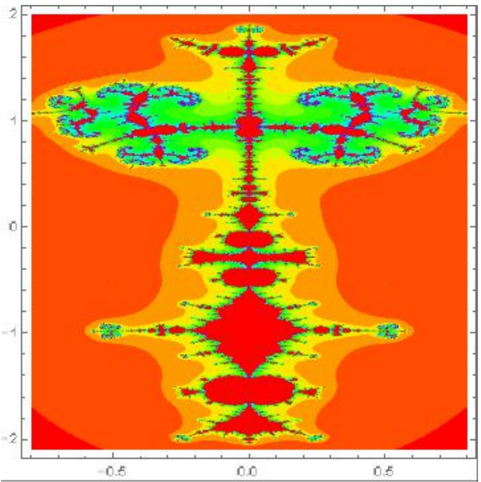

Figure 9: Antenna Cubic Julia set for u= v=.15 w=.83 m=2.5 n=-.80I

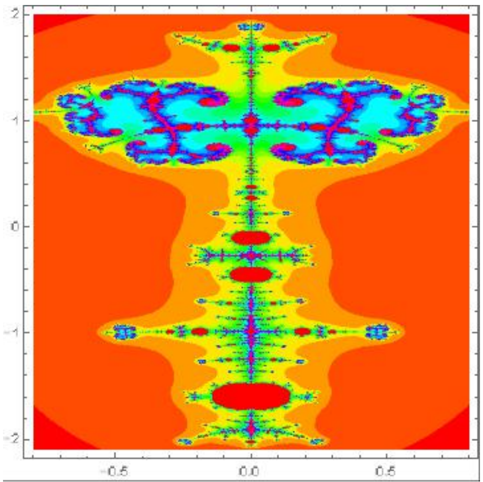

Figure 10: Antenna Cubic Julia set for u= v=.15 w=.83 m=2.62 n=-.80I

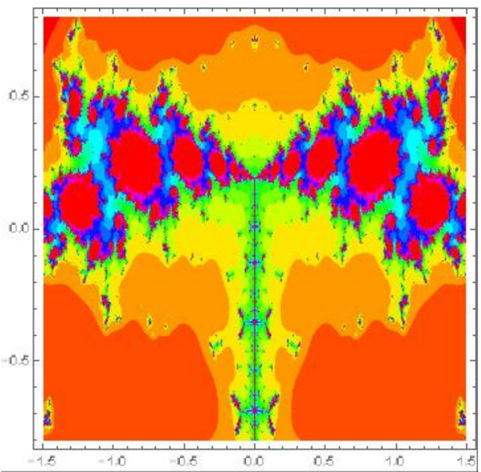

Figure 11: Antenna Cubic Julia set for u= v=.46 w=.366 m=-29.1 n=.75I

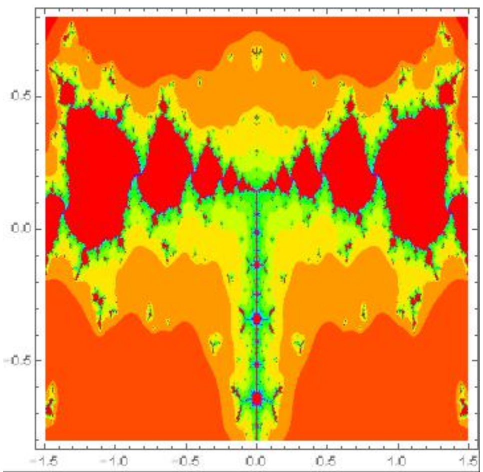

Figure 12: Antenna Cubic Julia set for u= v=.46 w=.366 m=-3.1 n=.59I

8

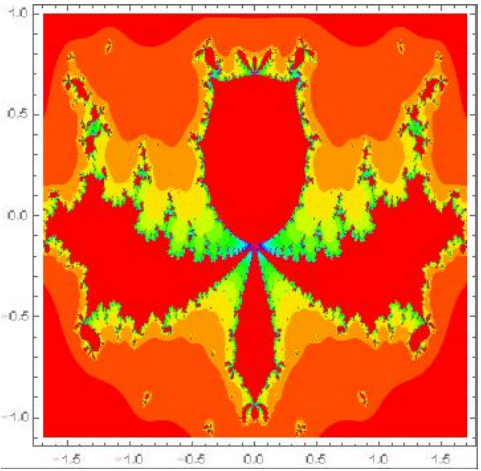

Figure 13: Cubic Julia set for u= v=.48,w=.32 m=-3.13, n=-.59I

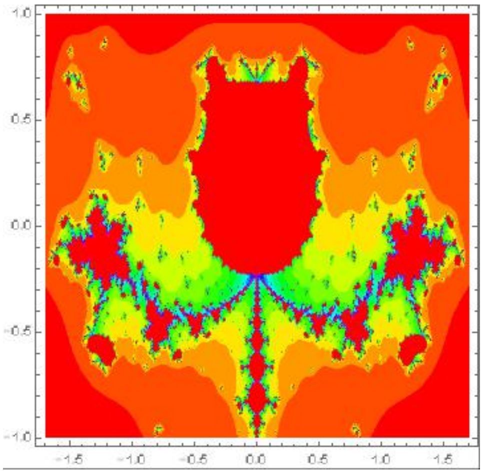

Figure 14: Cubic Julia set for u= v=.48, w=.32 m=-3.03, n=-.89I

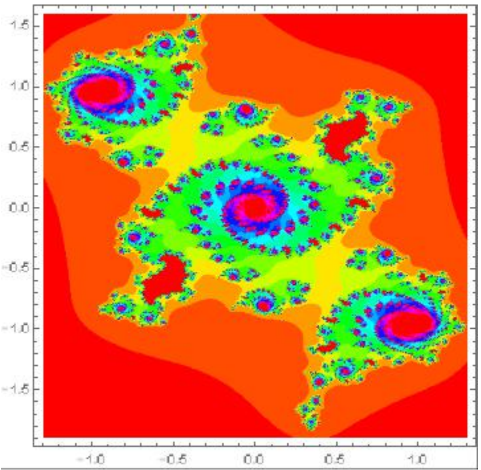

Figure 15: Cubic Julia set for u= v=.5, w=.03, m=1.2I n=0

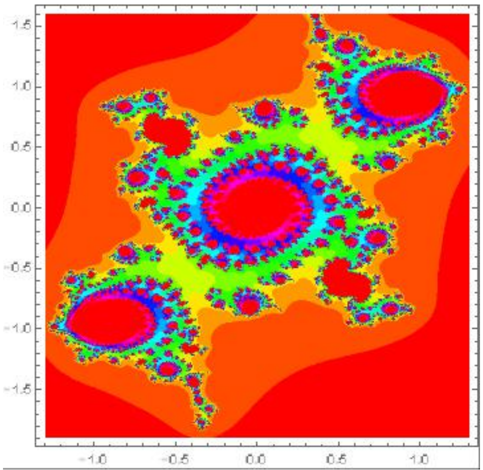

Figure 16: Cubic Julia set for u= v=.5, w=.05,m=-1.17I n=0

- It has been observed that the Figures 1 - 14 and Figures 17 - 20 show the perfect mathematical reflexive symmetry about imaginary axes only whereas the Figures 15 - 16 show the perfect mathematical reflexive symmetry about both the axes (real as well as imaginary axes).
- It is noticed that the prisoner sets of the cubic Julia sets in the Figures 1 - 5, Figure 17 and Figures 11 - 16 are mathematically disconnected whereas the prisoner sets of the remaining cubic Julia sets are mathematically connected.

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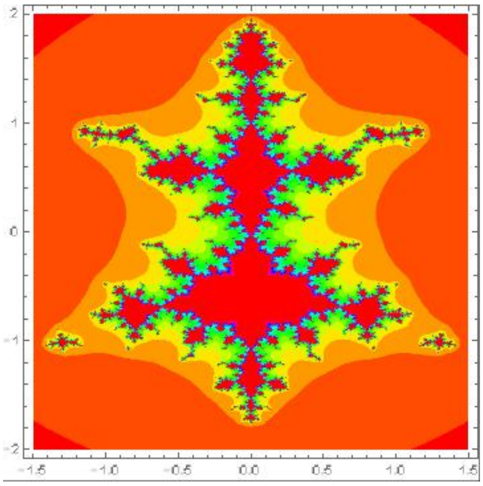

Figure 17: Cubic Julia set for u= v=.03 w=.95 m=1.0 n=1.001I

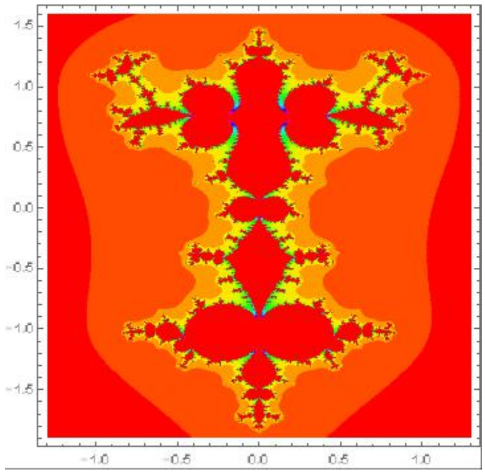

Figure 18: Cubic Julia set for u= v=.5 w=.01 m=1.3 n=-.80I

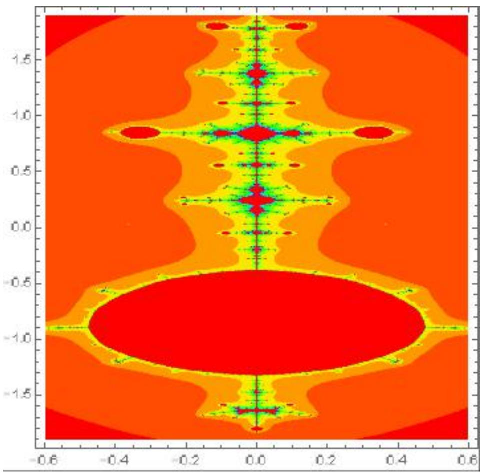

Figure 19: Cubic Julia set for u= v=.19 w=.376 m=2.2 n=.40I

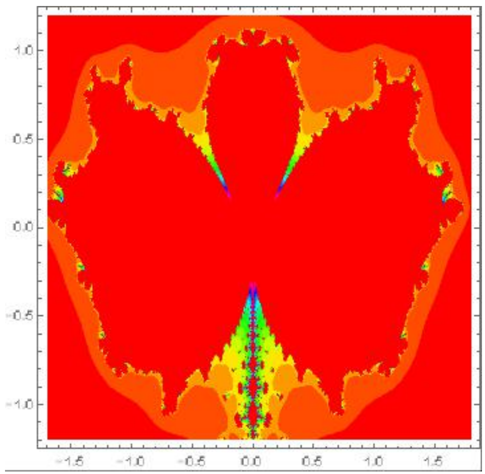

Figure 20: Butterfly Cubic Julia set for u= v=.5 w=.3 m=-2.03 n=-.6I

- It is also noticed that the cubic Julia sets in Figure 1 and Figure 2 capture the shape of ants, Figures 4 - 8, look like as meditating posture, the cubic Julia in Figures 9-12 and Figure 19 take the shape of antennas, the cubic Julia set in Figure 20 take the shape of a butterfly and the Julia sets in Figure 3-4 and Figures 13-14 are wall-decorated pictures.

Conclusion

In this research paper, a new different four-step iterative procedure has been applied to the complex cubic polynomial $P_{m,n}(z) = z^3 + mz + n$, and significant as well as exciting results have been obtained.We have obtained new and ever seen cubic Julia sets as output for the above complex cubic polynomial. Some of the generated cubic Julia sets have perfect mathematical reflexive symmetry about the axes. From above generated

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cubic Julia sets, some Julia sets are connected and some are disconnected which is also an important mathematical property of fractals.It is also fascinating to see that some of the generated cubic Julia sets take the shape of antennas which is an important application of fractals, some cubic Julia sets capture the shape of ants and the butterfly, some cubic Julia sets look like as meditating posture, and some other look like as wall-decorated pictures.

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The complement on the existence of fixed points that belong to the zero set of a certain function due to Karapinar et al.

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Abstract

Recently, the idea of φ-fixed point and the elementary results on φ-fixed points were first investigated by Jleli et al. [Jleli M, Samet B, Vetro C (2014) Fixed point theory in partial metric spaces via φ-fixed point's concept in metric spaces. Journal of Inequalities and Applications, 2014(1):1-9.]. Based on this work, Karapinar et al. [Karapinar E, O'Regan D, Samet B (2015) On the existence of fixed points that belong to the zero set of a certain function. Fixed Point Theory and Applications, 2015(1):1-14.] established the new φ-fixed point results, which can be reduced to the famous fixed point result of Boyd and Wong in 1969. However, the main result of Karapinar et al. does not cover the φ-fixed point results of Jleli et al. This paper aims to fulfill this gap by proving φ-fixed point results covering several φ-fixed point results and fixed point results. **Key words**: φ-fixed point; φ-Picard mapping; Control function
Mathematics Subject Classification(2010): 46T99; 47H10; 54H25

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1 Introduction and preliminaries

In 2014, Jleli et al. [1] had initiated the concept of (F, φ)-contraction with the help of some control function, which is one of the interesting generalizations of the classical Banach contraction principle and first introduced the concepts of φ-fixed point and φ-Picard mapping. Moreover, they also proved some φ-fixed point theorems for contractive mappings expanded some fixed point results in metric spaces. Consistent with Jleli et al. [1], we will be needed the following notations, definitions, and results in this research.

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Let X be a nonempty set, $\varphi : X \to [0,\infty)$ be a given function and $T : X \to X$ be a mapping. By F_T and Z_φ the set of all fixed points of T and the set of all zeros of the function φ, respectively, i.e., $F_T := \{x \in X : Tx = x\}$ and $Z_\varphi := \{x \in X : \varphi(x) = 0\}$.

Definition 1.1 ([1])**.** Let X be a nonempty set and $\varphi : X \to [0,\infty)$ be a given function. An element $z \in X$ is said to be a φ-fixed point of the mapping $T : X \to X$ if and only if z is a fixed point of T and $\varphi(z) = 0$ (i.e., $z \in F_T \cap Z_\varphi$).

Definition 1.2 ([1])**.** Let (X, d) be a metric space and $\varphi : X \to [0,\infty)$ be a given function. A mapping $T : X \to X$ is said to be a φ-Picard mapping if and only if, for each $x, z \in X$, the following conditions are satisfied:

(i) $F_T \cap Z_\varphi = \{z\}$ for some $z \in X$;

(ii) $T^n x \to z$ as $n \to \infty$ for each $x \in X$.

To describe the control function, which is an important class of this work, let $\mathcal{F}$ be the family of all functions $F : [0,\infty)^3 \to [0,\infty)$ satisfying the following conditions:

$(F1)$ $\max\{a, b\} \le F(a, b, c)$ for all $a, b, c \in [0,\infty)$;

$(F2)$ $F(0, 0, 0) = 0$;

$(F3)$ F is continuous.

As examples, the following functions $F_1, F_2, F_3 : [0,\infty)^3 \to [0,\infty)$ belong to $\mathcal{F}$:

(i) $F_1(a, b, c) = a + b + c$ for all $a, b, c \in [0,\infty)$;

(ii) $F_2(a, b, c) = \max\{a, b\} + c$ for all $a, b, c \in [0,\infty)$;

(iii) $F_3(a, b, c) = a + a^2 + b + c$ for all $a, b, c \in [0,\infty)$.

Definition 1.3 ([1])**.** Let (X, d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function, and $F \in \mathcal{F}$. A mapping $T : X \to X$ is said to be an (F, φ)-contraction mapping if there exists $k \in [0, 1)$ such that

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \le kF(d(x, y), \varphi(x), \varphi(y)) \quad \forall (x, y) \in X^2. \tag{1.1}$$

Theorem 1.4 ([1])**.** *Let (X, d) be a complete metric space, $\varphi : X \to [0,\infty)$ be a lower semi-continuous function, $F \in \mathcal{F}$ and $T : X \to X$ be an (F, φ)-contraction mapping. Then $F_T \subseteq Z_\varphi$ and T is a φ-Picard mapping.*

Remark 1.5. Note that if we set $F(a, b, c) = a + b + c$ for all $a, b, c \in [0,\infty)$ and $\varphi(x) = 0$ for all $x \in X$ in (1.1), then the contractive condition (1.1) reduces to the Banach contractive condition.

In recent years, Jleli et al.'s fixed point theorem has been generalized and extended in several directions. One such generalization was introduced by Karapinar et al. [2] by replacing the constant k of the contractive condition (1.1) with the control function, which was first introduced by Boyd and Wong [3]. They also proved the existence and uniqueness results of a φ-fixed point for new nonlinear mappings. Nevertheless, this result expands all conditions of results of [1], except that the condition $(F2)$ is replaced by

$(F2^{\star})$ $F(a,0,0)=a$ for all $a\geq 0$.

Here, we recall the definition of the following class as given by Boyd and Wong [3]. Denote Ψ the set of functions $\psi:[0,\infty)\to[0,\infty)$ satisfying the following conditions:

$(\psi 1)$ ψ is upper semi-continuous from the right;

$(\psi 2)$ $\psi(t)<t$ for each $t>0$.

Combining this definition with Jleli et al.'s theorem, Karapinar et al. [2] proved the following theorem:

Theorem 1.6 ([2])**.** *Let (X,d) be a complete metric space. Suppose that the mapping $T:X\to X$ satisfies the following condition:*

$$F(d(Tx,Ty),\varphi(Tx),\varphi(Ty))\leq\psi(F(d(x,y),\varphi(x),\varphi(y)))\quad\forall(x,y)\in X^2,\quad(1.2)$$

where $\varphi:X\to[0,\infty)$ is lower semi-continuous, $\psi\in\Psi$, and $F:[0,\infty)^3\to[0,\infty)$ is a function satisfying the following conditions:

$(F1)$ $\max\{a,b\}\leq F(a,b,c)$ *for all* $a,b,c\in[0,\infty)$*;*

$(F2^{\star})$ $F(a,0,0)=a$ *for all* $a\geq 0$*;*

$(F3)$ F *is continuous.*

Then $F_T\subseteq Z_{\varphi}$ and T is a φ-Picard mapping.

In the case of ψ defined by $\psi(t)=kt$ for some $k\in[0,1)$, Theorem 1.6 seem almost similar to a generalization of Theorem 1.4 except that Theorem 1.6 use the control function F satisfying conditions $(F1)$, $(F2^{\star})$, $(F3)$ rather than Theorem 1.4 use the control function F satisfying conditions $(F1)$, $(F2)$, $(F3)$. It is easy to see that the condition $(F2^{\star})$ is stronger than the condition $(F2)$ since there are many functions satisfying the condition $(F2)$ but it does not satisfy the condition $(F2^{\star})$. For example, functions $F_1,F_2,F_3:[0,\infty)^3\to[0,\infty)$ defined by $F_1(a,b,c)=a+a^2+b+c$, $F_2(a,b,c)=\ln(a+1)+(a+b)e^c+\max\{a,b\}$, and $F_3(a,b,c)=\max\{2a,b\}+c$ for all $a,b,c\geq 0$. From the above observation, we can conclude that the main theorem of [2] is not a proper extension of Theorem 1.4.

The main goal of this work is to fulfill the mentioned gap by using the new technique for improving Theorem 1.6 via the original control function, which was introduced by Jleli et al. in [1]. For simplicity, the following diagram shows the relation of Karapinar et al.'s results and our results, which describes the objectives of this research.

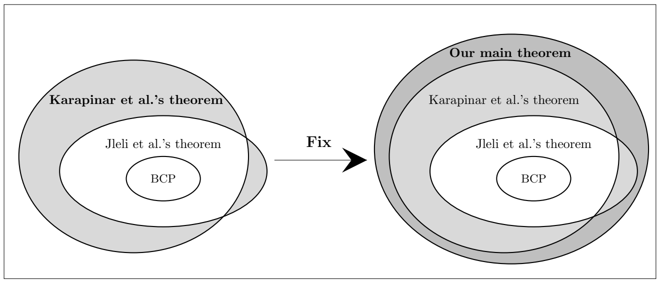

Figure 1: The conceptual research framework

2 Main results

In section, we will prove the generalized φ-fixed point results by using the new technique, which is the improved version of the φ-fixed point theorem of Karapinar et al. [2], but it replaces the condition $(F2^{\star})$ by the condition $(F2)$.

Theorem 2.1. *Let (X,d) be a complete metric space and T be a self mapping on X such that*

$$F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \leq \psi(F(d(x,y),\varphi(x),\varphi(y))) \quad \forall (x,y) \in X^2, \tag{2.1}$$

where $\varphi : X \to [0,\infty)$ is lower semi-continuous, $F \in \mathcal{F}$ and $\psi \in \Psi$. Then $F_T \subseteq Z_\varphi$ and T is a φ-Picard mapping.

Proof. The first step is to prove that $F_T \subseteq Z_\varphi$. Let $x \in F_T$. Letting $y = x$ in (2.1), we have

$$F(0,\varphi(x),\varphi(x)) \leq \psi(F(0,\varphi(x),\varphi(x))). \tag{2.2}$$

Assume that $\varphi(x) > 0$. It follows from $(F1)$ that $F(0,\varphi(x),\varphi(x)) > 0$. By (2.2) and $(\psi 1)$, we get

$$F(0,\varphi(x),\varphi(x)) \leq \psi(F(0,\varphi(x),\varphi(x))) < F(0,\varphi(x),\varphi(x)),$$

which is a contradiction. Therefore, $\varphi(x) = 0$, which implies that

$$F_T \subseteq Z_\varphi. \tag{2.3}$$

Next, we will show that T is a φ-Picard mapping. Let x_0 be an arbitrary point in X. Define the sequence $\{x_n\} \subseteq X$ by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n^*} = x_{n^*-1}$ for some $n^* \in \mathbb{N}$, then x_{n^*} is a fixed point of T. Hence, for the rest of the proof, we assume that $x_n = x_{n-1}$ for all $n \in \mathbb{N}$, that is,

$$d(x_n, x_{n-1}) > 0 \tag{2.4}$$

for each $n \in \mathbb{N}$. Now, we will claim that

$$\lim_{n\to\infty} d(x_{n+1},x_n) = \lim_{n\to\infty} \varphi(x_n) = 0. \tag{2.5}$$

From $(F1)$ and (2.4), we obtain

$$F(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1})) > 0$$

for all $n \in \mathbb{N}$. This allows to use the condition $(\psi 2)$ and so by using the contractive condition (2.1), we obtain

$$\begin{aligned} F(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)) &\leq \psi(F(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1}))) \\ &< F(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1})) \end{aligned} \tag{2.6}$$

for all $n \in \mathbb{N}$. This shows that $\{F(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n))\}$ is a decreasing sequence. Furthermore, it is easy to see that it is also bounded below by 0 and hence it converges to some point $r \geq 0$, that is,

$$\lim_{n\to\infty} F(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)) = r. \tag{2.7}$$

From (2.6), (2.7) and the squeeze theorem, we get

$$\lim_{n\to\infty} \psi(F(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1}))) = r. \tag{2.8}$$

Assume that $r > 0$. So we have

$$\begin{aligned} r &\overset{(2.8)}{=} \limsup_{n\to\infty} \psi(F(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1}))) \\ &\overset{(\psi 1)}{\leq} \psi(r) \\ &\overset{(\psi 2)}{<} r \end{aligned}$$

which provides a contradiction. Therefore, $r = 0$, that is,

$$\lim_{n\to\infty} F(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)) = 0,$$

and thus, by $(F1)$, we get

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = \lim_{n\to\infty} \varphi(x_n) = 0,$$

that is, Equation (2.5) holds.

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. Assume on the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) \geq k$ and

$$d\left(x_{m(k)}, x_{n(k)}\right) \geq \epsilon \tag{2.9}$$

for all $k \in \mathbb{N}$. Corresponding to $m(k)$, we may choose $n(k)$ such that it is the smallest integer satisfying (2.9). Then we have

$$d\left(x_{m(k)}, x_{n(k)-1}\right) < \epsilon.$$

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By the triangular inequality, we have

$$\begin{aligned}\epsilon &\leq d\left(x_{m(k)}, x_{n(k)}\right)\\ &\leq d\left(x_{m(k)}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{n(k)}\right)\\ &< \epsilon + d\left(x_{n(k)-1}, x_{n(k)}\right).\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (2.5), we have

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \epsilon. \tag{2.10}$$

By a similar way, we can show that

$$\lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) = \epsilon. \tag{2.11}$$

Using $(F3)$, (2.5), (2.10) and (2.11), it follows that

$$\lim_{k\to\infty} F\left(d\left(x_{m(k)}, x_{n(k)}\right), \varphi\left(x_{m(k)}\right), \varphi\left(x_{n(k)}\right)\right) = F(\epsilon, 0, 0) \tag{2.12}$$

and

$$\lim_{k\to\infty} F\left(d\left(x_{m(k)+1}, x_{n(k)+1}\right), \varphi\left(x_{m(k)+1}\right), \varphi\left(x_{n(k)+1}\right)\right) = F(\epsilon, 0, 0). \tag{2.13}$$

Now, we choose $x = x_{m(k)}$ and $y = x_{n(k)}$ in (2.1), we infer

$$F\left(d\left(x_{m(k)+1}, x_{n(k)+1}\right), \varphi\left(x_{m(k)+1}\right), \varphi\left(x_{n(k)+1}\right)\right) \leq \psi\left(F\left(d(x_{m(k)}, x_{n(k)}), \varphi(x_{m(k)}), \varphi(x_{n(k)})\right)\right).$$

Taking the limit superior as $k \to \infty$ on both sides of the above inequality and using (2.13), we deduce

$$F(\epsilon, 0, 0) \leq \limsup_{k\to\infty} \psi\left(F\left(d(x_{m(k)}, x_{n(k)}), \varphi(x_{m(k)}), \varphi(x_{n(k)})\right)\right). \tag{2.14}$$

Using the condition $(\psi 1)$ and (2.12), we obtain

$$\limsup_{k\to\infty} \psi\left(F\left(d(x_{m(k)}, x_{n(k)}), \varphi(x_{m(k)}), \varphi(x_{n(k)})\right)\right) \leq \psi(F(\epsilon, 0, 0)) < F(\epsilon, 0, 0). \tag{2.15}$$

From (2.14) and (2.15) together with $(\psi 2)$, we obtain

$$F(\epsilon, 0, 0) \leq \psi(F(\epsilon, 0, 0)) < F(\epsilon, 0, 0),$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there exists a point $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0. \tag{2.16}$$

Using (2.5), (2.16) and the lower semi-continuity of φ, we get

$$0 \leq \varphi(z) \leq \liminf_{n\to\infty} \varphi(x_n) = 0,$$

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which implies that

$$\varphi(z) = 0. \tag{2.17}$$

Next, we will prove that z is a fixed point of T. From $(F2)$, $(F3)$, (2.5) and (2.16), we get

$$\lim_{n\to\infty} F(d(x_n, z), \varphi(x_n), 0) = F(0,0,0) = 0.$$

Note that from $(\psi 2)$, it follows that $\lim_{t\to 0^+} \psi(t) = 0$. Then

$$\lim_{n\to\infty} \psi(F(d(x_n, z), \varphi(x_n), 0)) = \lim_{t\to 0^+} \psi(t) = 0. \tag{2.18}$$

Hence, from $(F1)$, (2.1), (2.17) and (2.18), we conclude that

$$d(x_{n+1}, Tz) \leq \psi(F(d(x_n, z), \varphi(x_n), 0)) \to 0 \text{ as } n \to \infty.$$

Therefore, by the uniqueness of the limit, we obtain $z = Tz$, i.e., z is a fixed point of T.

Finally, we will show that T has a unique fixed point. Suppose that u and v are fixed points of T such that $u \neq v$. Then $d(u,v) > 0$. Therefore,

$$\begin{aligned} F(d(u,v),0,0) &= F(d(Tu,Tv),0,0) \\ &\overset{(2.1)}{\leq} \psi(F(d(u,v),0,0)) \\ &\overset{(\psi 2)}{<} F(d(u,v),0,0), \end{aligned}$$

which is a contradiction. Thus, the fixed point of T is unique. This completes the proof. □

The following example shows that Theorem 2.1 is more applicable than many other results in the literature.

Example 2.2. Let $X = [0,\infty)$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x,y) = |x-y|$ for all $x, y \in X$. Then (X,d) is a complete metric space. Assume that $T : X \to X$ and $\psi : [0,\infty) \to [0,\infty)$ are defined by

$$Tx = \begin{cases} \frac{x^2}{2}, & 0 \leq x < \frac{1}{2}, \\ \frac{1}{8x}, & x \geq \frac{1}{2}, \end{cases} \quad \text{and} \quad \psi(t) = \begin{cases} \frac{t}{2}, & 0 \leq t < 1, \\ \frac{1}{2}\sin\left(\frac{1}{2t-1}\right) + \frac{1}{2}, & t \geq 1. \end{cases}$$

Clearly, by the graph in Figure 2, we have $\psi \in \Psi$.

Now, we will show that the fixed point result of Boyd and Wong [3] can not be applied in this example. For any $x \in \left(0, \frac{1}{2}\right)$ and $y = \frac{1}{2}$, we obtain

$$d(Tx, Ty) = \left|\frac{x^2}{2} - \frac{1}{4}\right| = \frac{1}{4} - \frac{x^2}{2} > \frac{1}{4} - \frac{x}{2} = \left|\frac{x}{2} - \frac{1}{4}\right| = \psi\left(\left|x - \frac{1}{2}\right|\right) = \psi\left(d(x,y)\right).$$

Hence, T does not satisfy the Boyd and Wong's contractive condition. Also, the Banach contraction principle is not applicable, since T is not continuous at $\frac{1}{2}$.

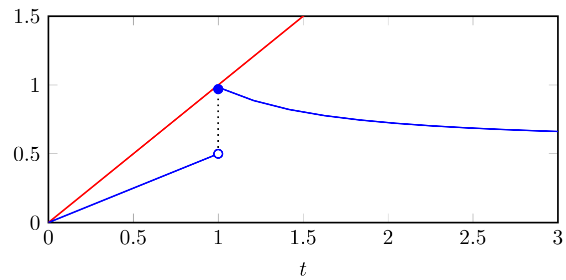

Figure 2: The graph of ψ in blue line

Next, we will show that Theorem 2.1 can be applied in this example. Let $\varphi : X \to [0, \infty)$ and $F : [0, \infty)^3 \to [0, \infty)$ be defined by

$$\varphi(x) = x, \quad x \in X \qquad \text{and} \quad F(a, b, c) = a + a^2 + b + c, \quad a, b, c \geq 0.$$

It is easy to see that $F \in \mathcal{F}$ and φ is lower semi-continuous. Now, we claim that the mapping T satisfies the contractive condition (2.1). Suppose that $x, y \in X$. We have to consider the following cases:

Case 1. If $(x, y) \in \left[0, \frac{1}{2}\right)^2$, then we get

$$\begin{aligned}
F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &= d(Tx,Ty) + (d(Tx,Ty))^2 + \varphi(Tx) + \varphi(Ty) \\
&= |Tx - Ty| + |Tx - Ty|^2 + Tx + Ty \\
&= \frac{|x^2 - y^2|}{2} + \frac{|x^2 - y^2|^2}{4} + \frac{x^2}{2} + \frac{y^2}{2} \\
&= \frac{|(x+y)(x-y)|}{2} + \frac{|(x+y)(x-y)|^2}{4} + \frac{x^2}{2} + \frac{y^2}{2} \\
&\leq \frac{|x-y|}{2} + \frac{|x-y|^2}{2} + \frac{x}{2} + \frac{y}{2} \qquad (2.19) \\
&\leq \psi(|x-y| + |x-y|^2 + x + y) \\
&= \psi(d(x,y) + (d(x,y))^2 + \varphi(x) + \varphi(y)) \\
&= \psi(F(d(x,y),\varphi(x),\varphi(y))).
\end{aligned}$$

Case 2. If $(x,y) \in \left[\frac{1}{2},\infty\right)^2$, then we get

$$\begin{aligned}F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &= d(Tx,Ty) + (d(Tx,Ty))^2 + \varphi(Tx) + \varphi(Ty)\\ &= |Tx-Ty| + |Tx-Ty|^2 + Tx + Ty\\ &= \left|\frac{1}{8x} - \frac{1}{8y}\right| + \left|\frac{1}{8x} - \frac{1}{8y}\right|^2 + \frac{1}{8x} + \frac{1}{8y}\\ &< \frac{1}{2}\sin\left(\frac{1}{2(|x-y|+|x-y|^2+x+y)+1}\right) + \frac{9}{16} \qquad (2.20)\\ &= \psi(|x-y|+|x-y|^2+x+y)\\ &= \psi(d(x,y) + (d(x,y))^2 + \varphi(x) + \varphi(y))\\ &= \psi(F(d(x,y),\varphi(x),\varphi(y))).\end{aligned}$$

Case 3. Let $(x,y) \in \left[0,\frac{1}{2}\right) \times \left[\frac{1}{2},\infty\right) \cup \left[\frac{1}{2},\infty\right) \times \left[0,\frac{1}{2}\right)$. Without loss of generality, we may assume that $x \in \left[0,\frac{1}{2}\right)$ and $y \in \left[\frac{1}{2},\infty\right)$ and so $|x-y|+|x-y|^2+x+y > 1$, then

$$\begin{aligned}F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &= d(Tx,Ty) + (d(Tx,Ty))^2 + \varphi(Tx) + \varphi(Ty)\\ &= |Tx-Ty| + |Tx-Ty|^2 + Tx + Ty\\ &= \left|\frac{x^2}{2} - \frac{1}{8y}\right| + \left|\frac{x^2}{2} - \frac{1}{8y}\right|^2 + \frac{x^2}{2} + \frac{1}{8y}\\ &\le \frac{1}{2}\sin\left(\frac{1}{2(|x-y|+|x-y|^2+x+y)+1}\right) + \frac{9}{16} \qquad (2.21)\\ &= \psi(|x-y|+|x-y|^2+x+y)\\ &= \psi(d(x,y) + (d(x,y))^2 + \varphi(x) + \varphi(y))\\ &= \psi(F(d(x,y),\varphi(x),\varphi(y))).\end{aligned}$$

The validity of the conditions (2.19), (2.20) and (2.21) can be checked by plotting 3D surface in MATLAB, shown as Figure 3. Without loss of generality and for the sake of simplicity, we restrict the domain in Figure 3 to $[0,3]$. Therefore, all the required hypotheses of Theorem 2.1 are fulfilled, and so T has a unique φ-fixed point. In this case, the point 0 is a unique φ-fixed point of T.

Remark 2.3. If we take $\varphi(x) = 0$ for all $x \in X$ in Theorem 2.1, then we get the real proper generalization of the Boyd and Wong fixed point theorem. However, if we take the same function φ in Theorem 1.6 and use $(F2^{\star})$, we can see that the obtained result is equivalent to the Boyd and Wong fixed point theorem. This yields the advantage of our main result with the several results in the literature as shown in Figure 4.

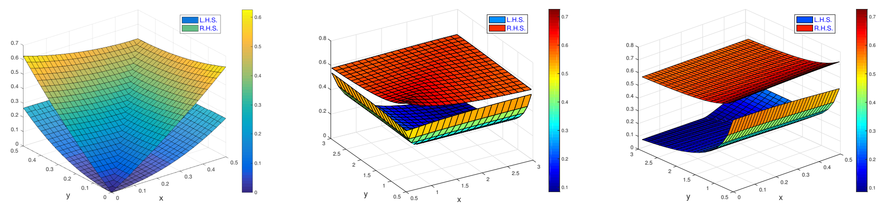

Figure 3: The value of the comparison of the L.H.S. and the R.H.S. of (2.19) and (2.21)

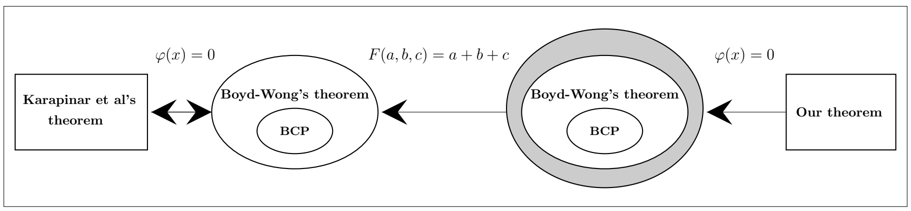

Figure 4: The difference of consequence between our theorem and Karapinar et al.'s theorem

3 Conclusions

Inspired by the problem of the relaxing of the hypothesis of the control function F in Theorem 1.6, we proposed a new technique for solving this problem. By the help of this suggested technique, our main theorem has the new proof, which seems to be simpler than the proof in [2]. The obtained result of this paper is a real proper generalization of the result in [1], and it also covers several famous fixed point results and φ-fixed point results in the literature. For the part of an application, we can use the main result in this work for applying in the homotopy result, and the fixed point results in partial metric spaces like the application in [2] since the class $\mathcal{F}$ is weaker than the class defined in [2].

Acknowledgement

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Dynamical Analysis on Two Dose Vaccines in the Presence of Media

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Abstract

The Covid-19 outbreak hit us as a serious health crisis with vaccination to be seen as the only way out of it. And media can play the role of an advocate to fight against this epidemic by spreading important awareness regarding various protocols and vaccination strategies. Since breakthrough infections are becoming highly common, a two dose vaccine regime may be the need of the hour even for individuals with a pre-existing illness to build better immunity. Thus in this paper, our aim is to analyse a mathematical model with two dose vaccination strategy in the presence of media and breakthrough infections. An SIV_1V_2R model is formulated and the dynamical analysis is done. The basic reproduction number is obtained and the local stability analysis of both the disease-free and endemic equilibrium point is discussed based on reproduction number. The global stability of the endemic equilibrium point is done by graph theoretic approach. Finally, the numerical validation of the analytic solution is done using MATLAB using the real data of India for some important parameters to address a few vital questions which involves the role of media on the vaccination strategy. And sensitive model parameters effecting the basic reproduction number and endemic equilibrium points are identified using sensitivity analysis techniques like PRCC(partial rank correlation coefficient). Thus, the outcomes demonstrated the trend a two-dose vaccine model can follow and how the effect of media can help bring down the infections. This model provides support that two dose vaccination against COVID-19 with media presence for awareness is highly effective in controlling this epidemic.

Keywords: Vaccine; Covid-19; Global Stability; Parameter Sensitivity

1 Introduction

Since the advancements in medication and technology particularly with the initiation of vaccination, there has been an improvement in the quality of life

1

in the age of infectious diseases. After the primary creation of vaccine by a doctor for pox from a live pox virus in 1796, in order to produce vaccines for alternative diseases several scientists and doctors followed his path as well. Many diseases are currently preventable like contagious disease, mumps, rubella, serum hepatitis, and respiratory illness due to the use of such vaccines[1, 2]. Due to the spread of SARS-CoV-2 virus, the COVID-19 outbreak was declared a pandemic and since then the development of COVID-19 vaccines has been the top priority of responsible stakeholders to control the outbreak. Afterwards with the availability of various types of vaccines to the world, the focus shifted to process of vaccination. There is an urgent need for the disbursement of vaccines to the general public so that the vaccines are effective to suppress the infection and in a very timely manner. Therefore, coming up with concerned strategies is crucial to the success of vaccination and control of epidemic as several of the vaccines need over one dose over a lifespan. Specially since the talk has now been shifted on the number of doses(one or two) [3], decisions regarding single or multi dose vaccines is a matter of great importance to avoid confusion among the general public.

1.1 Two-Dose Vaccine

For the ongoing Covid-19 epidemic, the often suggested vaccine strategy may be a two-dose program and even a booster dose in future[4, 5]. The second dose isn't thought of to be a booster vaccine but rather the use of this particular dose is with the aim to produce immunity to those that don't answer the primary dose. For example, more or less two to five of individuals don't develop immunity once the primary dose of the MMR(Measles, Mumps and Rubella) vaccine is given emphasising on the need of multi dose regime. Recently Covid-19 Vaccines have received emergency use authorization developed by Oxford-AstraZeneca, Pfizer-BioNTech and Moderna in different countries. Mass vaccination campaigns and clinical trials can provide high levels of protection against severe and symptomatic disease using 2 dose vaccines [6]. Due to a weak one-dose vaccine immunity in some vaccines, there could be short-term benefits where the virus may still continue to replicate [7]. This could eventually lead to immune escape mutations by the virus and thus a two-dose strategy may be able to mitigate this effect. Even for individuals who already have some existing illness, when co-infected with Covid may show better immunity when administered by two dose of vaccines than one dose vaccine as seen in the case of cancer patients in [8]. Multidose vaccines when compared to single-dose injection may offer a stronger protection against infection of the same vaccines and communication initiatives are needed to spread information about such regimes[9]. The ongoing discussions related to vaccination regimes are often led by media and influences the decision making of individuals.

1.2 Effect of media

The behavior of public with respect to vaccination may be altered due to involvement of media. People who may be infected may not come in contact with others because of the weakening effects due to their illness or due to the suggestions by public health organizations to quarantine to avoid infecting others. Hygienic measures may be taken up by general public to reduce the chance of getting infected and take steps to avoid large public gatherings. An example is of the 1994 outbreak of plague which presented with complex dynamics in a state in India [10]. After the outbreak of the disease many people in order to escape the disease fled the state of Surat and led to the transmission of the disease to other parts of the country. Thus, there is a need of proper discourse of information to the public. The media in particular greatly influences an individual's behavior toward a diseases and may also lead to interventions to control disease spread by governmental health care institutions. Awareness programs by media can make people comprehensive about a disease towards taking precautionary measures like wearing protective masks, social distancing and more importantly vaccination to suppress the chances of infection.

1.3 Empirical Literature and Structure of study

The most recent development in mathematical modelling in the field of biology or epidemic can be seen in [11, 12, 13, 14, 15]. There has been innumerable developments in mathematical modeling and numerical methods and its applications which is able to provide a better understanding and prediction for various types of systems like models depicting the relationship between computer viruses and epidemiology [16, 17, 18, 19, 20]. Mathematical models are able to provide a compatible understanding with the real-world dynamics of infection diseases. In order to exhibit the dynamics of Covid-19, there are many models available in the literatures for systems of nonlinear differential equations, making the models more realistic [21]. We have seen good researches in epidemic or infectious disease models[22, 23]. In [24], a deterministic model for Varicella Zoster Virus dynamics with vaccination is studied. Mathematical model on the outbreaks of influenza and to manage it by vaccination is discussed in [25]. A Dengue Epidemic model is considered amid vaccination in [26] and in [27] dynamic models is discussed with the importance of vaccination. And as of the recent Covid-19 outbreak some models with respect to vaccinations are discussed are discussed in [28, 29]. In [7] one dose regime is recommend if it produces a strong immune response. However, if a single vaccine dose is poor then the manufacturer recommended two-dose regime is suggested for a potential positive long term outcomes. Thus, a two dose vaccine Covid-19 model needs to be studies to understand its impact on the transmission of infection.

There may be some countries like the developing nations that may not be able to sustain a two-dose vaccination program for respiratory illness, and definitely would not be able to get funding for the multi-dose respiratory illness inoculation process. Thus, one needs to address the following questions: is it

doable to form one dose respiratory illness vaccination program that would replace a two-dose respiratory illness vaccination strategy? Is the involvement of media important in increasing vaccination process and reducing infection?. We shall aim to address these questions by developing a multi-dose vaccine model consisting of the susceptible, infected, vaccinated(First and second dose) and recovered (SIV_1V_2R) individuals in Section 2 and investigating the impact of media involvement to dispense information to the public. In section 3 the model dynamics are analyzed for the equilibrium point. We shall establish the local and global(graph theoretic method) for the endemic equilibrium. In Section 5 we will proceed with the numerical simulation where in we shall valid our results and understand the behaviour of our system. Under numerical discuss, we aim to find the sensitivity Indices of endemic equilibrium point to find the relevant parameters and their impact on the populations, followed by uncertainty analysis for the basic reproduction number to find important parameters related to transmission of infection in a two dose regime system. As part of our study, numerical discussion will help quantify the sensitivity index of the various parameters and give an insight to understand the effectiveness of the two dose regime and media to our variables and transmission of infection.
The novelty of our study is to encapsulate a two dose vaccination regime and the role of the media for a Covid-19 system and dynamically analysing thoroughly along with real data numerical validation.

2 The Model

The Model developed in this paper is motivated by the model by Kermack and McKendrick [30] which consists of the Susceptible, Infected and Removed (SIR) epidemiological class. SIR model was one of the revolutionary in its time but in present life with full of advanced technology, SIR model is one of the cornerstones of Mathematical Epidemiology. While assuming constant birth and death rate, SIR model divided the population into three different classes; Susceptible(S),Infected(I) and Recovered (R). The working of the SIR model can bee seen in Fig 1 for better understanding.

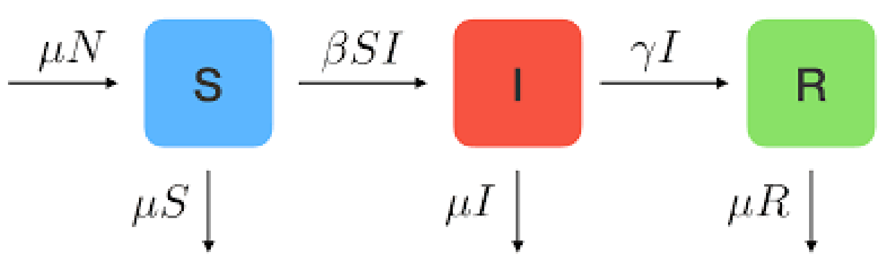

Figure 1: Flowchart of Model

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The differential Equations for the basic **SIR** model is ans follows:

$$\frac{dS}{dt} = \mu N - \beta SI - \mu S$$
$$\frac{dI}{dt} = \beta SI - \gamma I - \mu I$$
$$\frac{dR}{dt} = \gamma I - \mu R.$$

A two-dose regime may be able to provide better immunity to the general public and even to those who have some pre-existing illness [8]. Thus with this as motivation, we have extended the paper by incorporating two new vaccinated classes V_1 and V_2 in reference to the current scenario of covid. The model assumptions are considered as follows:

- Only a fraction of susceptible population get vaccinated due to the rumours regarding vaccinations.
- The interaction between susceptible and infected classes follow Holling type-II functional response.
- The population can still join the susceptible class and be prone to getting infected after two doses. These kind of infections are termed as 'Breakthrough' infections [31, 32] and exist for all types of vaccines prescribed against SARS-COVID-2. Breakthrough infections can be attributed to occurrence of severe variants (such as the delta variant), low immune response to vaccination and traveling to places that are seeing significant surge in cases. But the infections are mild in nature and may not lead to hospitalisation.
- The natural recovery is also not permanent and they can still get reinfected as defined by Indian Council Of Medical Research (ICMR). ICMR defines this reinfection as occurrence of two positive tests at a gap of at least 102 days with one interim negative test [31] .

Therefore, in reference to the assumptions, the extended model SIV_1V_2R is given by,

$$\begin{aligned}
\frac{dS}{dt} &= (1-p)\mu N - \mu S - \frac{b_1 IS}{(1+\alpha I)} + \psi R \\
\frac{dI}{dt} &= \frac{b_1 IS}{(1+\alpha I)} - \mu I - \gamma I \\
\frac{dV_1}{dt} &= p\mu N - \mu V_1 - p_1 V_1 \\
\frac{dV_2}{dt} &= p_1 V_1 - \mu V_2 - p_2 V_2 \\
\frac{dR}{dt} &= \gamma I + p_2 V_2 - \mu R - \psi R.
\end{aligned} \tag{1}$$

5

The system is bounded in the region $\{S, I, V_1, V_2, R; S + I + V_1 + V_2 + R = N\}$.

Table 1: Table for Variables and Parameters

Variables and Parameters	**Interpretation**
S	Susceptible Individual Density
I	Infected Individual Density
R	Recovered Individual Density
V_1	Vaccinated Individual Density After 1st Dose
V_2	Vaccinated Individual Density After 2nd Dose
N	Total Population Density
μ	Birth and Death rate
p	Rate of First Dose of Vaccine
p_1	Rate of Second Dose of Vaccine
p_2	Rate at which Vaccinated Individuals get Recovered
b_1	Rate of Infection
γ	Rate at which Infected Individuals Recover/ Natural Recovery Rate
ψ	Rate at which Recovered Individuals get Susceptible Again
α	Effect of Media

The description of the parameters and variables can be seen in Table 1 for our system. We can see the mechanism graphically in the schematic diagram in Fig 2 for the proposed model.

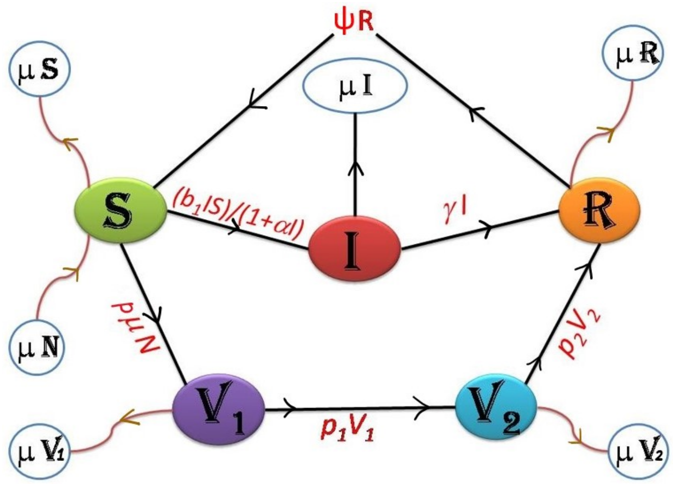

Figure 2: Schematic diagram of the SIV_1V_2R model. Susceptible individuals S can either move to the infected I or the vaccinated class V_1. At the rate of p, the susceptible who get vaccinated with the first dose will join vaccinated class V_1 . After recovering naturally the infected individuals join the recovered class R at the rate of γ. At the rate of p_1 individuals who receive the second dose of vaccination after getting the first dose will move towards the vaccinated class V_2. Individuals move towards the recovered class R at the rate of p_2 after receiving both the doses. Individuals from the recovered class R join back to the class of susceptible S at the rate of ψ owing to breakthrough infections/reinfections even after after getting vaccinated with both the doses or recovering from the infection naturally.

2.0.1 Questions Addressed by the Research Work

Our research takes a dig at the following unaddressed issues:

- Exploring the calibre of infected class response to complete vaccination. Can vaccination act as a lodestar to reduce infection ?
- The relation of media and pandemic. What is the role of media in infected and susceptible classes ?
- What is the degree of correlation, if there exist any, between the parametric values and the endemic equilibrium?.

3 Model Dynamics

3.1 Basic Reproduction Number

The basic reproduction number particularly for the study infectious diseased is considered a central concept and is the spectral radius of the next generation

method. For the dynamic analysis of any general disease transmission model, the basic reproduction number R_0 [33, 34] is a crucial element. The trend or behaviour of R_0 can give significant implications for upcoming outbreaks. It gives the number of secondary infections arising due to a single infection. In the system, if $R_0 > 1$ then the disease will continue and if $R_0 \leq 1$ then the disease will die out. To explore the impact of vaccination we will be dealing with $R_0 > 1$ when the infection is present in the system. R_0 can be written as,

$$R_0 = \frac{b_1 N}{(\mu + \gamma)}.$$

3.2 Existence of equilibrium points

Equilibrium points are the steady state solutions where the system may approach in the long run. Our analysis will be around the equilibrium points and its stability as it can help to study the system behaviour in long run in reference to multi-dose vaccination. Thus, we shall first obtain the equilibrium points and the conditions for their existence. Our main focus will be on disease-free equilibrium point and endemic equilibrium point [35]. The system (1) posses a disease free or boundary equilibrium point $E^0(S^0, 0, V_1^0, V_2^0, R^0)$ given by,

$$S^0 = \frac{(1-p)N(\mu+\psi)(\mu+p_1)(\mu+p_2) + pp_1p_2\psi N}{(\mu+\psi)(\mu+p_1)(\mu+p_2)}, \quad V_1^0 = \frac{p\mu N}{(\mu+p_1)}$$

$$V_2^0 = \frac{pp_1\mu N}{(\mu+p_1)(\mu+p_2)}, \quad R^0 = \frac{pp_1p_2\mu N}{(\mu+\psi)(\mu+p_1)(\mu+p_2)}.$$

Endemic or interior equilibrium point $E^*(S^*, I^*, V_1^*, V_2^*, R^*)$ for the system (1) is given by

$$S^* = \frac{(\mu+\gamma)(1+\alpha I^*)}{b_1} = \frac{(1+\alpha I^*)N}{R_0},$$

$$V_1^* = \frac{p\mu N}{(\mu+p_1)}, \quad V_2^* = \frac{pp_1\mu N}{(\mu+p_1)(\mu+p_2)},$$

$$I^* = \frac{\mu(\mu+\psi)(\mu+p_1)(\mu+p_2)[(1-p)Nb_1 - (\mu+\gamma)] + \psi b_1 pp_1p_2\mu N}{[(\mu+\psi)(\mu+\gamma)(\alpha\mu+b_1) - \psi b_1\gamma](\mu+p_1)(\mu+p_2)},$$

$$R^* = \frac{\gamma\mu(\mu+\psi)(\mu+p_1)(\mu+p_2)[(1-p)Nb_1 - (\mu+\gamma)] + \psi b_1 pp_1p_2\mu N\gamma + pp_1p_2\mu N[(\mu+\gamma)(\mu+\psi)(\alpha\mu+b_1-\psi b_1\gamma)]}{(\mu+\psi)(\mu+p_1)(\mu+p_2)[(\mu+\psi)(\mu+\gamma)(\alpha\mu+b_1) - \psi b_1\gamma]},$$

where the equilibria exists if $R_0 > \frac{1}{(1-p)}$ and $(\alpha\mu + b_1) > \psi b_1\gamma$.

We will now be analysing the stability of boundary and interior equilibrium points for the system (1).

3.3 Local Stability analysis

We shall prove the local stability for the model about the disease free and endemic equilibrium point to visualize the conditions under which the epidemic system can be stabilized [36]. General Jacobian matrix for our system is given by,

$$J = \begin{vmatrix} (-\mu - \frac{b_1 I}{1+\alpha I}) & -\frac{b_1 S}{(1+\alpha I)^2} & 0 & 0 & \psi \\ \frac{b_1 I}{(1+\alpha I)} & (\frac{b_1 S}{(1+\alpha I)^2} - \mu - \gamma) & 0 & 0 & 0 \\ 0 & 0 & (-\mu - p_1) & 0 & 0 \\ 0 & 0 & p_1 & (-\mu - p_2) & 0 \\ 0 & \gamma & 0 & p_2 & (-\mu - \psi). \end{vmatrix}$$

General characteristic equation pertaining to the jacobian matrix above is given by,

$$\left[\left(\mu + \frac{b_1 I}{1+\alpha I} + \lambda\right)\left(\frac{b_1 S}{(1+\alpha I)^2} - \mu - \gamma - \lambda\right)(\mu + p_1 + \lambda)(\mu + p_2 + \lambda)(\mu + \psi + \lambda)\right],$$

$$\left[\left(-\frac{b_1 I}{1+\alpha I}\right)\left(\frac{b_1 S}{(1+\alpha I)^2}\right)(\mu + p_1 + \lambda)(\mu + p_2 + \lambda)(\mu + \psi + \lambda)\right]\left[\left(\frac{b_1 I\gamma\psi}{1+\alpha I}\right)(\mu + p_1 + \lambda)(\mu + p_2 + \lambda)\right] = 0.$$

Characteristic equation pertaining to the boundary equilibrium point E^0 is given by,

$$(\mu + \lambda)(b_1 S^0 - \mu - \gamma - \lambda)(\mu + p_1 + \lambda)(\mu + p_2 + \lambda)(\mu + \psi + \lambda) = 0.$$

Eigen values corresponding to boundary equilibrium point E^0 are $\lambda_1 = -\mu, \lambda_2 = b_1 S^0 - (\mu + \gamma) < 0$ if $b_1 S^0 < (\mu + \gamma)$ or $R_0 < \frac{N}{S^0}, \lambda_3 = -(\mu + p_1), \lambda_4 = -(\mu + p_2), \lambda_5 = -(\mu + \psi)$. Consequently, E^0 is stable if $R_0 < \frac{N}{S^0}$. Next, the characteristic equation pertaining to interior equilibrium point $E^*(S^*, I^*, V_1^*, V_2^*, R^*)$ is given as follows:

$$\lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5 = 0,$$

where
$a_1 = 5\mu + \psi + p_1 + p_2 + \gamma + \frac{b_1 I}{1+\alpha I} - \frac{b_1 S}{(1+\alpha I)^2}$,
$a_2 = -(\mu + \frac{b_1 I}{1+\alpha I})(\frac{b_1 S}{(1+\alpha I)^2} - \mu - \gamma) + \frac{b_1^2 SI}{(1+\alpha I)^3} + (3\mu + \psi + p_1 + p_2)(2\mu + \gamma + \frac{b_1 I}{1+\alpha I} - \frac{b_1 S}{(1+\alpha I)^2}) + (\mu + \psi)(2\mu + p_1 + p_2) + (\mu + p_1)(\mu + p_2)$,
$a_3 = (3\mu + \psi + p_1 + p_2)\{\frac{b_1^2 SI}{(1+\alpha I)^3} - (\mu + \frac{b_1 I}{1+\alpha I})(\frac{b_1 S}{(1+\alpha I)^2} - \mu - \gamma)\} + [(2\mu + p_1 + p_2)(\mu + \psi) + (\mu + p_1)(\mu + p_2)][2\mu + \gamma + \frac{b_1 I}{1+\alpha I} - \frac{b_1 S}{(1+\alpha I)^2}] + (\mu + \psi)(\mu + p_1)(\mu + p_2) - \frac{b_1 I\psi\gamma}{1+\alpha I}$,
$a_4 = \{(2\mu + p_1 + p_2)(\mu + \psi) + (\mu + p_1)(\mu + p_2)\}\{\frac{b_1^2 SI}{(1+\alpha I)^3} - (\mu + \frac{b_1 I}{1+\alpha I})(\frac{b_1 S}{(1+\alpha I)^2} - \mu - \gamma)\} + (\mu + \psi)(\mu + p_1)(\mu + p_2)[2\mu + \gamma + \frac{b_1 I}{1+\alpha I} - \frac{b_1 S}{(1+\alpha I)^2}] - (2\mu + p_1 + p_2)\frac{b_1 I\psi\gamma}{1+\alpha I}$,
$a_5 = (\mu + \psi)(\mu + p_1)(\mu + p_2)[\frac{b_1^2 SI}{(1+\alpha I)^3} - (\mu + \frac{b_1 I}{1+\alpha I})(\frac{b_1 S}{(1+\alpha I)^2} - \mu - \gamma)] - (\mu + p_1)(\mu + p_2)\frac{b_1 I\psi\gamma}{1+\alpha I}$.
Thus, the endemic equiliria is locally stable according to Routh- Hurwitz criteria

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if $a_i's > 0, i = 1,2,3,4,5$ under the following conditions: $I^*(1+\alpha I^*) > S^*$, $R_0 > \frac{N(1+\alpha I^*)^2}{S^*}$, $\frac{b_1^2 S^* I^*}{(1+\alpha I^*)^3} > (\mu + \frac{b_1 I^*}{1+\alpha I^*})(\frac{b_1 S^*}{(1+\alpha I^*)^2} - \mu - \gamma)$, $\frac{I^*}{1+\alpha I^*} < \frac{(\mu+\psi)(\mu+p_1)(\mu+p_2)}{b_1\psi\gamma}$, $(\mu+\psi)(\mu+p_1)(\mu+p_2)[2\mu+\gamma+\frac{b_1 I^*}{1+\alpha I^*} - \frac{b_1 S^*}{(1+\alpha I^*)^2}] > (2\mu+p_1+p_2)\frac{b_1 I^*\psi\gamma}{1+\alpha I^*}$ and $(\mu+\psi)[\frac{b_1^2 S^* I^*}{(1+\alpha I^*)^3} - (\mu + \frac{b_1 I^*}{1+\alpha I^*})(\frac{b_1 S^*}{(1+\alpha I^*)^2} - \mu - \gamma)] > \frac{b_1 I^* \psi\gamma}{1+\alpha I^*}$ along with $a_1a_2a_3 > a_3^2 + a_1^2 a_4$ and $(a_1a_4 - a_5)(a_1a_2a_3 - a_3^2 - a_1^2a_4) > a_5(a_1a_2 - a_3)^2 + a_5^2 a_1$.

4 Global stability

To establish global stability we shall use the graph-theoretic method as in [37, 38, 39]. We shall construct the lyapunov function through a directed graph with the help of the terminologies from [37]. To construct the lyapunov function we shall use a directed graph . A directed graph has a set of ordered pair say (i,j) and vertices where (i,j) is known as arc to terminal vertex j from initial vertex i. For the terminal vertex j, $d^-(j)$ is the in-degree of j which denotes the number of arcs in the digraph. And for initial vertex is i, $d^+(i)$ is the out-degree of vertex i which denotes the number of arcs in the digraph. Let us consider a weighted directed graph say $\chi(J)$ over a $q \times q$ weighted matrix J where the weights(a_{ij}) of each arc if they exist are $a_{ij} > 0$, and if otherwise then $a_{ij} = 0$. we consider c_i as the co-factor of l_{ij} of the Laplacian of $\chi(J)$ which is given by:

$$l_{ij} = \begin{cases} -a_{ij} & i \neq j \\ \sum_{k\neq i} a_{ij} & i = j. \end{cases}$$

If there is a strongly connected path i.e directed to and fro path for the arcs in $\chi(J)$ then $c_i > 0 \forall i = 1,2...,q$. We shall also use Theorem 3.3 and Theorem 3.4 from [37, 39], which will help us in the construction of lyapunov function. The theorems states:

- **Theorem3.3 of [37]**: if $a_{ij} > 0$ and $d^-(i) = 1$, for some i, j, then
$$c_i a_{ij} = \sum_{k=1}^{q} c_j a_{jk}.$$
- **Theorem 3.4 of [37]**: if $a_{ij} > 0$ and $d^+(j) = 1$, for some i, j, then
$$c_i a_{ij} = \sum_{k=1}^{q} c_k a_{ki.}$$

We shall also use **Theorem 3.5** of [37] as below:

Theorem 4.1. *Let us consider an open set $L \subset R^m$ and a function $f : L \to R^m$ for a system*

$$\dot{z} = f(z) \tag{2}$$

and assuming:
a) $\exists\ M_i : L \to R$, $H_{ij} : L \to R$ and $a_{ij} \geq 0$ such that

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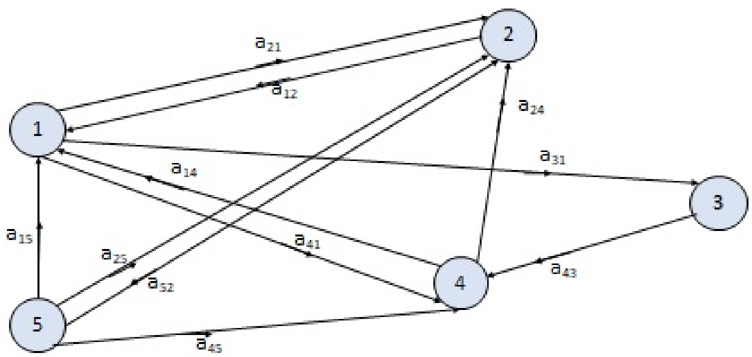

Figure 3: Directed graph for $\alpha = 1$ for the associated weights

$$M_i' = M_i'|_2 \leq \sum_{j=1}^{q} a_{ij} H_{ij}(z),\ with\ z \in L,\ i = 1, \ldots, q.$$

b) For $J = [a_{ij}]$, of (H, J) each directed cycle B_c satisfies:

$$\sum_{(i,j)\in\epsilon(B_c)} H_{ij}(z) \leq 0\ ,\ z \in L$$

where $\epsilon(B_c)$ is set of arcs in B_c.
Then for $c_i \geq 0, i = 1, \ldots, q$ the function is:

$$M(z) = \sum_{i=1}^{q} c_i M_i(z)$$

satisfies $M'|_2 \leq 0$, that is, $M(z)$ is a Lyapunov function for 2.

4.1 Lyapunov Function Construction

Construction: $M_1 = \frac{(S-S^*)^2}{2}$, $M_2 = I - I^* - I^* \ln \frac{I}{I^*}$, $M_3 = \frac{(V_1-V_1^*)^2}{2}$, $M_4 = \frac{(V_2-V_2^*)^2}{2}$ and $M_5 = \frac{(R-R^*)^2}{2}$. Now by differentiation;
$M_1' = (S - S^*)S' \leq ((1-p)\mu N + \mu S^*)(S + V_2) + \frac{b_1 S^* IS}{(1+\alpha I)} + \psi RS = a_{14}H_{12} + a_{12}H_{12} + a_{15}H_{15}$ where $a_{14} = (1-p)\mu N + \mu S^*, a_{12} = b_1 S^*, a_{15} = \psi$.
$M_2' = (\frac{I-I^*}{I})I' \leq \frac{b_1 IS}{(1+\alpha I)} + (\mu+\gamma)I^*(I + V_2 + 1) = a_{21}H_{21} + a_{24}H_{24}$ where $a_{21} = b_1, a_{24} = (\mu+\gamma)I^*$.
$M_3' = (V_1 - V_1^*)V_1' \leq (p\mu N + \mu V_1^* + p_1 V_1^*)(V_1 + S) = a_{31}G_{31}$ where $a_{31} = (p\mu N + \mu V_1^* + p_1 V_1^*)$.
$M_4' = (V_2 - V_2^*)V_2' \leq p_1 V_1 V_2 + (\mu + p_2)(V_2 + S) = a_{43}G_{43} + a_{41}G_{41}$ where $a_{43} = p_1, a_{41} = (\mu + p_2)$.
$M_5' = (R - R^*)R' \leq \gamma IR + p_2 V_2 R + (\mu+\psi)(R + I) = a_{25}G_{25} + a_{45}G_{45} + a_{52}G_{52}$ where $a_{25} = \gamma, a_{45} = p_2, a_{52} = (\mu+\psi)$. We get an associated weighted directed graph as shown in Fig 3. Then by Theorem3.5[37] $\exists c_i's, 1 \leq i \leq 5$ such that $M = \sum_{i=1}^{q} c_i M_i$ is a lyapunov function. Using Theorem 3.3 and 3.4 we get the relation between c_i.

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For $a_{31} > 0$ and $d^-(3) = 1$, we get $c_3a_{31} = c_4a_{43}$ and for $a_{52} > 0$ and $d^-(5) = 1$ we get $c_5a_{52} = c_1a_{15} + c_2a_{25} + c_4a_{45}$. Hence, $c_1 = c_4 = c_2 = 1$, $c_3 = \frac{a_{43}}{a_{31}}$ and $c_5 = \frac{a_{15}+a_{25}+a_{45}}{a_{52}}$. Thus, the lyapunov function is $M = M_1 + M_2 + \frac{p_1}{(p\mu N+\mu V_1^*+p_1V_1^*)}M_3 + M_4 + \frac{p_2+\gamma+\psi}{(\mu+\psi)}M_5$. And for M':
$M' = (S - S^*)S' + (\frac{I-I^*}{I})I' + \frac{p_1}{(p\mu N+\mu V_1^*+p_1V_1^*)}(V_1 - V_1^*)V_1' + (V_2 - V_2^*)V_2' + \frac{p_2+\gamma+\psi}{(\mu+\psi)}(R - R^*)R'$.
If we consider the set $U = \{x \in R_+^5 : M' = 0\}$ then we see that $S = S^*$, $V_1 = V_1^*$, $V_2 = V_2^*$, $I = I^*$ and $R = R^*$. Hence, we get the unique equilibrium point $(S^*, I^*, V_1^*, V_2^*, R^*)$. Therefore we say that the equilibrium point is globally stable using LaSalle's Invariance principle .

5 Numerical Simulation

In this section, we will discuss a numerical example in support of the analytic results of our system. We would try to encapsulate the sensitivity analysis of endemic equilibrium, global sensitivity analysis of basic reproduction number along with validation of our analytic solution for media effect. So, on computing

Parameters	**Values/Units**	**Source**
μ	0.0035342	Assumed
p	0.004545	https://www.mygov.in/covid-19
p_1	0.0001	https://www.mygov.in/covid-19
p_2	0.00909	https://www.mygov.in/covid-19
b_1	0.62	Assumed
γ	0.0476	Assumed
ψ	0.0011	https://www.mygov.in/covid-19
α	0.5	Assumed
N	140	Assumed

Table 2: Parameters and Values for $\mathrm{SIV_1V_2R}$ model

the values using the parameters (mentioned in Table 2 where some values taken from the mygov.in site on 15 june,2021.) in the system of equations and we'll get unique positive equilibrium at (SIV_1V_2R) resting at (0.3348, 6.0267, 16.8756, 1.5341 ,25.3862) as seen in Fig 4, giving a glimpse about the behaviour or the condition of epidemic in future. Now, if we focus on the effect of vaccination on other classes like Susceptible and Infected, then we get to know by graph that:

- From Fig 5a, graph to analyse the relation between susceptible and Vaccinated class(First dose). Here we can see that the susceptible individuals are constant with increase in the vaccination process but on further increasing the vaccination, the susceptible individuals are exponentially increasing. As more and more people get vaccinated then through word of mouth and more confidence build on the idea of vaccination, more susceptible people will be willing to get vaccinated.

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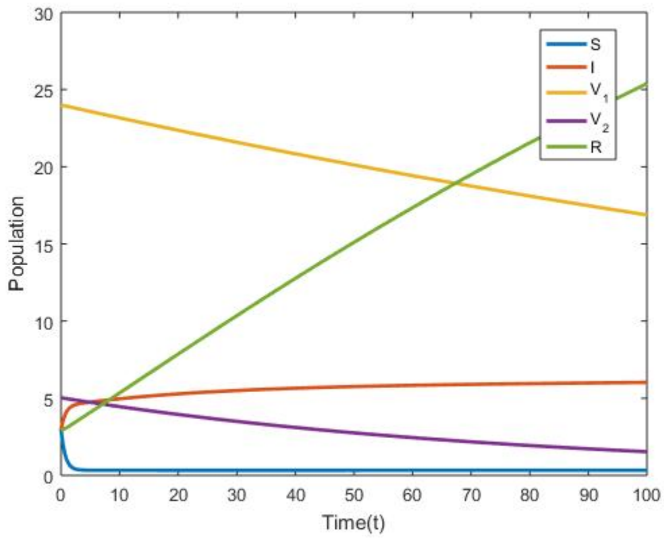

Figure 4: SIV_1V_2R Dynamic Graph

- Now by analysing the effect of complete(two dose) vaccination over infection rate (from Fig 5b), we get to know that the infectants goes on declining as individuals are getting vaccinated. This implies that two dose vaccination regime will help suppress transmission of infection and suppress the outbreak.

5.1 Sensitivity Indices of endemic equilibrium point

We shall now discuss the sensitivity analysis of equilibrium point with respect to the estimated parameters of our system. We aim to investigate the degree to which a parameter may affect the concerned variable through an affirmative relationship or a negative relationship through this process of parameter sensitivity analysis. We get the proportion that a relative change in a parameter brings to the relative change in a variable through the sensitivity index obtained. *Definition*[40, 41]: For the variable v that depends differentiably on a parameter h, we define the normalised forward sensitivity index Ω of a variable as:

$$\Omega_h^v = \frac{\partial v}{\partial h} \times \frac{h}{v}. \tag{3}$$

Thus, using the above method we get the sensitivity indices for each variable with respect to the parameters for endemic equilibrium as given in Table 3 and shown in Fig 6. For interpretation, if the index is positive it means that an increase in parameter will lead to the increase in the variable with the index value/magnitude. And a negative index implies that a increase in the parameter will lead to decrease in variable by the index value.

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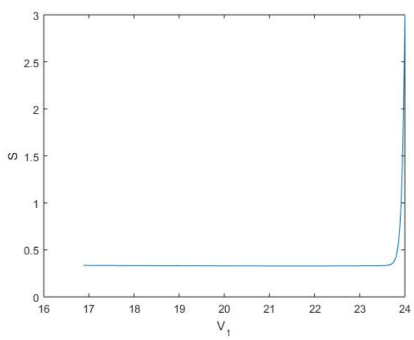
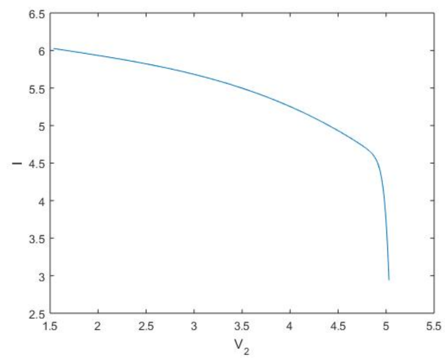


(a) Relation between Susceptible and Vaccinated classes



(b) Relation between Vaccinated and Infected classes

Figure 5: Relations with respect to Vaccinated class

-	S^*	I^*	V_1^*	V_2^*	R^*
p	-0.039	$-.0046$	1.1240	1.0837	-0.0096
α	-0.8494	0.0036	0	0	0.0042
p_1	$1.0794*10^{-4}$	$1.2664*10^{-4}$	-0.2222	0.7498	$6.6345*10^{-4}$
p_2	$3.9757*10^{-5}$	$4.7074*10^{-5}$	0	-0.7214	$2.4647*10^{-4}$
γ	0.1506	-0.9124	0	0	0.1019
μ	0.7197	0.7636	-0.7854	-1.0332	0.1196
ψ	0.1591	0.1867	0	0	-0.0219
N	0.8817	1.0461	1.1336	1.1475	1.2177
b_1	-0.9964	0.0042	0	0	0.0049

Table 3: Sensitivity indices, $\Omega^{v_i}_{h_j} = \frac{\partial v_i}{\partial h_j} * \frac{h_j}{v_i}$, of the state variables at the endemic equilibrium, v_i, to the parameters, h_j

5.2 Uncertainty analysis of R_0

For our model we shall also determine uncertainty analysis for R_0 by LHS method to get more validation for the relation between R_0 and its parameters. PRCC(partial rank correlation coefficient)[42] is one technique which will help us quantify the uncertainty for any model. We consider the four parameters from R_0 and have chosen normal distribution for them as in Fig 7. We find the PRCC values using Matlab with the following pdfs and shown in:
$b_1 \sim \quad Normal(0.62, 0.01)$,
$N \sim \quad Normal(140, 0.2)$,
$\gamma \sim \quad Normal(0.0476, 0.01)$,
$\mu \sim \quad Normal(0.0035342, 0.01)$.

We get the PRCC values for our input parameters which can be seen in Fig8. We get the following indexes for the parameters: $b_1 = 0.21, N = 0.26, \mu = -0.96$

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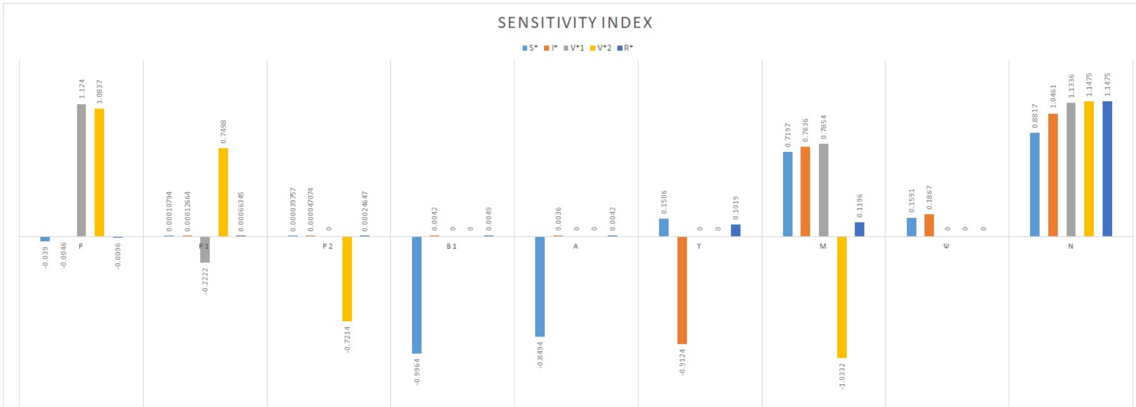


Figure 6: Bar graph for Sensitivity Index of $S^*, I^*, V_1^*, V_2^*, R^*$ with respect to the parameters

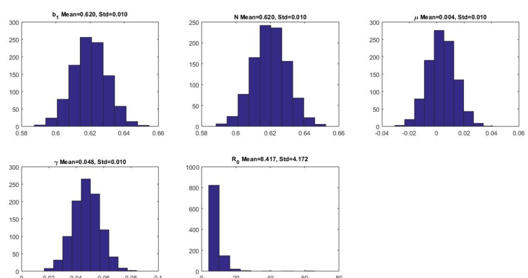


Figure 7: Distribution of parameters for R_0

and $\gamma = -0.96$. The graphs shows that R_0 is positively correlated to b_1 and N. The effect of the parameter μ and γ will bring about an opposite change in the transmission of infection as it is negatively correlated. Since the value μ and γ parameters are close to 1, it indicates a strong correlation to change in R_0. Thus, these two parameters are strongly negatively correlated to R_0.
We shall now check the contour plot for R_0 with respect to some combination of important parameters as in Fig 9. In Fig 9a we can see that the b_1 has a direct response to R_0, as the value of b_1 increases the gradation of color approaches to the highest color which is yellow. In a similar manner in Fig 9b we see that N too has direct response as we can see the color gradation approach yellow as N increases. And μ has a high indirect response to R_0, as the value of μ increases the gradation of color approaches to the lowest color which is dark blue and R_0 decreases.

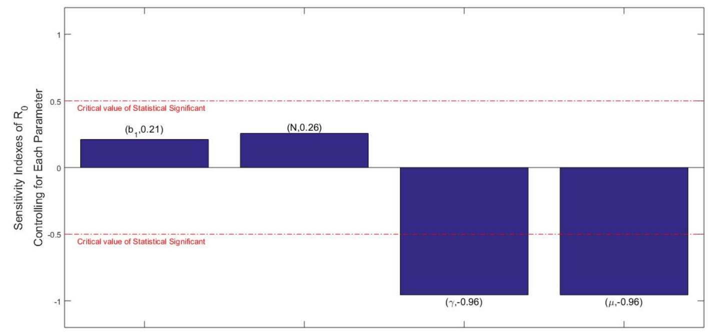
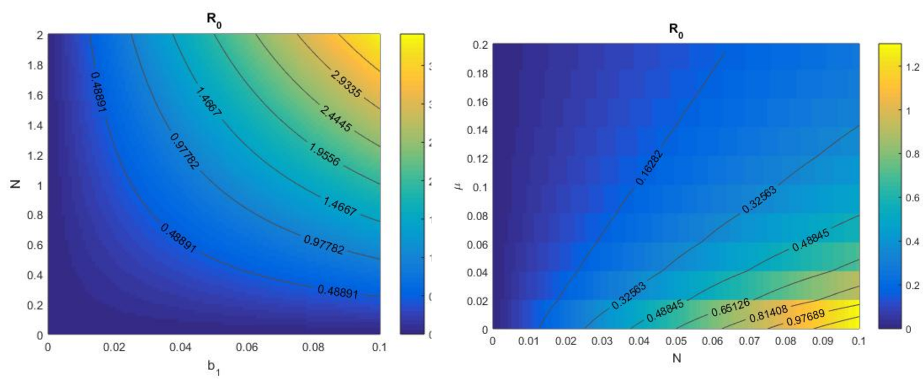


Figure 8: PRCC: input variables



(a)

(b)

Figure 9: Contour plots of ρ^0.

5.3 Media Effect (α) on Susceptible and Infected class

Media Effect helps in spreading awareness among people to stay safe from epidemic and has an ideal impact on the epidemic dynamics which can be seen through graph line when α is either at 0.5 or at 0 in the following:

- In Fig 10a, both values of α effect on susceptible population are shown. As the value of α increases, S also increases as now they will be less vulnerable to catching infection and avoiding decrease in S population. Thus, media will have a positive impact on S individuals as they get awareness of the implications of catching infections, advantages of vaccinations or protocols to follow to avoid risk of getting infected.
- We see in Fig 10b, that media causes a decreasing effect in infected class. Its shows that media effect plays an effective role in decreasing infection

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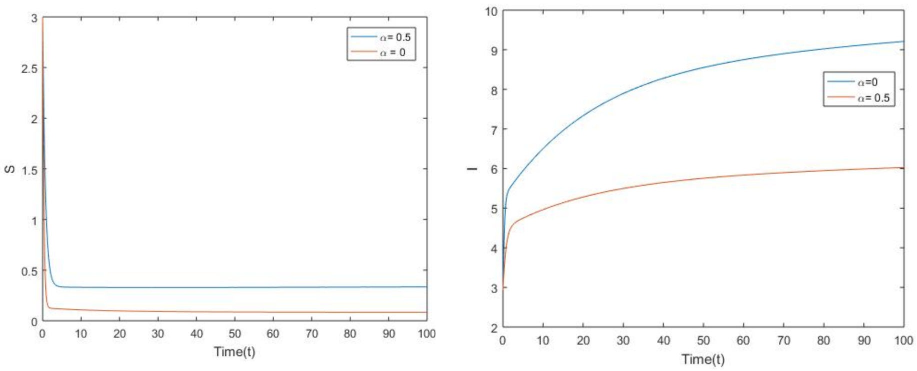
and spreading awareness among population about the epidemic outbreak.



(a) Media effect on Susceptible class

(b) Media effect on Infected class

Figure 10: Media Effect

6 Conclusion

In this paper our aim is to completely analyze the model and be able to become answerable to all those question we aimed to address at the start of our study. As novelty we have studied a two dose vaccination regime and the role of the media for a Covid-19 system and dynamically analysed our system thoroughly along with real data numerical validation. We have looked up on the analysis of model and studied their steady states like Disease Free Equilibrium(DFE) and Endemic Equilibrium(EE). We also found their local stability by finding their eigen values and using Routh-Hurwitz Stability Criteria. In context to the endemic equilibria, global stability analysis of the system has been performed using the graph-theoretic method. For numerical analysis we have taken some real data from Government site of India as an example of Covid-19 to make graph on them thus to analyse the situation created. The incorporation of two dose vaccination regime in the system brings about a desirable outcome for reducing infection. Later we used sensitivity analysis technique to identify sensitive model parameters effecting the R_0 and endemic equilibrium point. It was able to provide us with the degree of positive or negative correlation with which each variable is bound to the parameters present in the model. We have performed Latin hypercube sampling method for our model to determine uncertainty analysis for R_0 to understand the role certain parameters play in the transmission of infection. We also showed the effective role media plays in spreading awareness among population and help reduce infection. Also it is understood that α (the effect of media) will bring down the infections and increase the susceptible population by offering subsequent protection by awareness.

Therefore, on analysing we have seen that without media effect there is no such great awareness among people because media is one of the important ways that makes a country united or be aware of the various protocols and regimes related to an outbreak. Also, a two dose vaccination regime may be the need of the hour to vanquish such an infectious diseases. *As the cases of breakthrough infections and co-infections of already existing ailments increase, so the application of a two dose vaccine regime along with media influence can help provide better immunity and suppress infections as seen in our results.* Thus, a two dose vaccination regime and the role of the media is of paramount importance in policy formation and execution for this deadly epidemic.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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Certain problems in ordered partial metric space using mixed g-monotone

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Abstract

Motivated from the fixed point hypothesis, we demonstrate the presence and uniqueness for coupled coincidence type point including a contractive condition for a map in partially metric utilizing mixed g-monotone. A model is likewise outfitted to exhibit the legitimacy of the speculations of our outcomes.

Key words: Complete Metric Space; Coupled Fixed Point; mixed g-monotone property

Mathematics Subject Classification(2010): 54H25; 47H10

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1 Introduction

A different idea of generalized metric space perceived as partial metric space offered by Matthews [6]. Many authors had given imperative results on such type of spaces [8, 9, 10].

Bhaskar and Lakshmikantam [2] established coupled fixed point and demonstrated certain coupled fixed point results for maps which gratify the property of mixed monotone. Also, present applications for periodic boundary value problem. Authors have extended numerous outcomes on coupled fixed point hypotheses on metric spaces, e.g., in [1, 2, 3, 4, 5, 7, 11].

We foremost verify the presence of coupled coincidence points. Then, we demonstrate uniqueness of coupled coincidence point results for a map having the property of mixed g-monotone in partial metric spaces. At the end we support the result by giving an example.

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The view of partial metric spaces given by Matthews [6].

Definition 2.1. [6] Presuppose Z be a null set. A partial metric on Z defines as a function $p : Z \times Z \to \mathbb{R}_n$ for every $s, t, z \in Z$:

$$\begin{aligned} &(1) \quad s = t \iff p(s,s) = p(s,t) = p(t,t) \\ &(2) \quad p(s,s) \leq p(s,t), \\ &(3) \quad p(s,t) = p(t,s) \\ &(4) \quad p(s,t) \leq p(s,t) + p(z,t) - p(z,z) \end{aligned}$$

A pair (Z, p) known as partial metric space, and p is partial metric on Z where, Z is a null set.

If p is a partial metric on Z, the function $p^r : Z \times Z \to \mathbb{R}_+$ defined as

$$p^r(s,t) = 2\,p(s,t) - p(s,s) - p(t,t)$$

is a metric on Z.

The prevailing definitions given by [2].

Definition 2.2. [2] A point $(s, v) \in Z \times Z$ for a map $T : Z \times Z \to Z$ possesses $T(s, v) = s, T(v, s) = h$ then it is known as coupled fixed point.

Definition 2.3. [4] Assume $(S, \leq)$ is partially ordered set, let two mappings $F : S \times S \to S$ and $g : S \to S$. Then F possesses property of the mixed g-monotone if $F(s, w)$ is g-non-decreasing in its starting element and is g-non-increasing in its next element, for $s, w \in S$

$$\begin{aligned} s_1, s_2 \in S, gs_1 \leq gs_2 &\implies F(s_1, w) \leq F(s_2, w) \\ w_1, w_2 \in S, gw_1 \leq gw_2 &\implies F(s, w_1) \geq F(s, w_2). \end{aligned}$$

2 Main Theorem

Theorem 2.1. *Presuppose $(Z, \leq, p)$ be a complete partially ordered set. Presuppose mappings $T : Z \times Z \to Z$ and $g : Z \to Z$ possesses property of mixed g-monotone. Presuppose $T(Z \times Z) \subseteq g(Z)$ and for any $s_0, v_0 \in Z$ with $gs_0 \leq T(s_0, v_0)$ and $gv_0 \geq T(v_0, s_0)$*

$$\begin{aligned} p(T(s,v), T(u,z)) \leq p(gs, gu) - \psi(p(gv, gz)) \\ + L\, min\{p(gs, T(s,v)), p(gu, T(u,z)), \\ p(gu, T(s,v)), p(gs, T(u,z))\}. \end{aligned} \tag{2.1}$$

for every $s, u, v, z \in Z$ with $gs \geq gu$, $gv \leq gz$, and $L \geq 0$. Here ψ : [0,∞)→[0,∞) is a map which is non-decreasing, continuous and non-negative in (0,∞), ψ(0) = 0 and $\lim_{t\to\infty} \psi(t) = \infty$. Presuppose either Z has the subsequent properties or T is continuous.

1. *If a decreasing sequence $\{v_m\} \to Z$, therefore $gv_m \geq v$ for every m.*
2. *If an increasing sequence $\{s_m\} \to Z$, therefore $gs_m \leq s$ for every m.*

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Then T and g possesses coupled coincidence point.

Proof. Presuppose $s_0, v_0 \in Z$ such that $gs_0 \leq T(s_0, v_0)$ and $gv_0 \geq T(v_0, s_0)$. Since $T(Z \times Z) \subseteq g(Z)$, select $s_1, v_1 \in Z$ thus $gs_1 = T(s_0, v_0)$ and $gv_1 = T(v_0, z_0)$.

Again, take $s_2, v_2 \in Z$ such that $gs_2 = T(s_1, v_1)$ and $gv_2 = T(v_1, s_1)$. As T possesses the property of mixed g- monotone, we have $gs_0 \leq gs_1 \leq gs_2$ and $gv_2 \leq gv_1 \leq gv_0$. Persistent the same procedure, we can create $\{z_m\}$ and $\{v_m\}$ in Z such that

$$gs_m = T(s_{m-1}, v_{m-1}) \leq gs_{m+1} = T(s_m, v_m)$$

and

$$gv_{m+1} = T(v_m, s_m) \leq gv_m = T(v_{m-1}, s_{m-1}).$$

If, for some integer m, we have $(gs_{m+1}, gv_{m+1}) = (gs_m, gv_m)$, then $T(s_m, v_m) = gs_m$ and $T(v_m, s_m) = gv_m$,thus T and g has a coincidence point (s_m, v_m).
We presume that $(gs_{m+1}, gv_{m+1}) \neq (gs_m, gv_m)$ for all $m \in \mathbb{N}$, that is, we assume that either $gs_{m+1} \neq gs_m$ or $gv_{m+1} \neq gv_m$. we have,

$$\begin{aligned} p(gs_{m+1}, gs_m) &= p(T(s_m, v_m), T(s_{m-1}, v_{m-1})) \\ &\leq p(gs_m, gs_{m-1}) - \psi(p(gv_m, gv_{m-1})) \\ &+ L\ min\{p(gs_m, T(s_m, v_m)), p(gs_{m-1}, T(s_{m-1}, v_{m-1})), \\ &\qquad p(gs_m, T(gs_{m-1}, v_{m-1})), p(gs_{m-1}, T(s_m, v_m))\} \\ &= p(gs_m, gs_{m-1}) - \psi(p(gv_m, gv_{m-1})) \end{aligned} \tag{2.2}$$

similarly,

$$p(gv_{m+1}, gv_m) \leq p(gv_m, gv_{m-1}) - \psi(p(gs_m, gs_{m-1})) \tag{2.3}$$

Let $\delta_m = p(gs_{m+1}, gs_m) + p(gv_{m+1}, gv_m)$.
Add (2) and (3), we have

$$\delta_m \leq \delta_{m-1} - \psi(\delta_{m-1}) \tag{2.4}$$

If $\exists m_1 \in N^*$ s.t. $p(gs_{m_1}, gs_{m_1-1}) = 0, p(gv_{m_1}, gv_{m_1-1}) = 0$, then $gs_{m_1-1} = gs_{m_1} = T(gs_{m_1-1}, gv_{m_1-1}); gv_{m_1-1} = gv_{m_1} = T(gv_{m_1}, gs_{m_1-1})$ and T has coupled coincidence point and the evidence is done. In other case $p(gs_{m+1}, gs_m) \neq 0$; $p(gv_{m+1}, gv_m) \neq 0$ for every $m \in \mathbb{N}$. At that point utilizing presumption on ψ , we get

$$\delta_m \leq \delta_{m-1} - \psi(\delta_{m-1}) \leq \delta_{m-1} \tag{2.5}$$

δ_m is a positive sequence and possesses a limit δ^*. Take limit $m \to \infty$, we have

$$\delta^* \leq \delta^* - \psi(\delta^*)$$

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Thus $\psi(\delta^*) = 0$, utilizing supposition on ψ , we accomplished $\delta^* = 0$, ie. $\lim_{m\to\infty}(\delta_m) = 0$

$$\begin{aligned}\lim_{m\to\infty} p(s_{m+1}, s_m) + p(v_{m+1}, v_m) = 0\\ \implies \lim_{m\to\infty} p(s_{m+1}, s_m) = \lim_{m\to\infty} p(v_{m+1}, v_m) = 0\end{aligned} \tag{2.6}$$

We will show that $\{gs_m\}$, $\{gv_m\}$ are Cauchy groupings in Z. Assume that in any event one $\{gs_m\}$ or $\{gv_m\}$ be not a Cauchy sequence. At that point there exists $\epsilon > 0$ and two subsequence $m_k > n_k \geq k$ such that

$$r_k = p(gs_{m_k}, gs_{n_k}) + p(gv_{m_k}, gv_{n_k}) \geq \epsilon, \tag{2.7}$$

$\forall\ k = 1, 2, 3, \ldots$. Further, relating to n_k, select m_k such that it is smallest integer $m_k > n_k \geq k$ gratify (2.7), we have

$$p(gs_{m_k}, gs_{n_k}) + p(gv_{m_k}, gv_{n_k}) < \epsilon. \tag{2.8}$$

Using triangle inequality and (2.7) and (2.8), we get

$$\begin{aligned}\epsilon \leq r_k &= p(gs_{m_k}, gs_{n_k}) + p(gv_{m_k}, gv_{n_k})\\ &\leq p(gs_{m_k}, gs_{n-1_k}) + p(gs_{n-1_k}, gs_{n_k}) + p(gv_{m_k}, gv_{n-1_k}), p(gv_{n-1_k}, gv_{n_k})\\ &< \epsilon + \delta_{m_{k-1}}\end{aligned}$$

Let $k \to \infty$ and taking equation (2.6), we have $\lim_{n,m\to\infty} r_k = \epsilon > 0$.
Now, we get

$$\begin{aligned}p(gs_{m_{k+1}}, gs_{n_{k+1}}) &= p(T(gs_{m_k}, gv_{m_k}), T(gs_{n_k}, gv_{n_k}))\\ &\leq p(gs_{m_k}, gs_{n_k}) - \psi(p(gv_{m_k}, gv_{n_k}) + L\ min\{p(gs_{m_k}, T(gs_{m_k}, gv_{m_k})), p(gs_{n_k}, T(gs_{n_k}, gv_{n_k})),\\ &\qquad p(gs_{n_k}, T(gs_{n_k}, gv_{n_k})), p(gs_{m_k}, T(gs_{m_k}, gv_{m_k}))\}\\ &\leq p(gs_{m_k}, gs_{n_k}) - \psi(p(gv_{m_k}, gv_{n_k}).\end{aligned} \tag{2.9}$$

Similarly,

$$p(gv_{m_{k+1}}, gv_{n_{k+1}}) \leq p(gv_{m_k}, gv_{n_k}) - \psi(p(gs_{m_k}, gs_{n_k}). \tag{2.10}$$

Using (2.9) and (2.10), we get

$$r_{k+1} \leq r_k - \psi(r_k) \tag{2.11}$$

$\forall\ k \in 1, 2, 3, \ldots$ take $k \to \infty$ in equation (11).

$$\epsilon = \lim_{k\to\infty} r_{k+1} \leq \lim_{k\to\infty}[r_k - \psi(r_k)] < \epsilon. \tag{2.12}$$

a contraction. Thus $\{gs_m\}$ and $\{gv_m\}$ are Cauchy sequence.
Using lemma, $\{gs_m\}$ and $\{gv_m\}$ are Cauchy sequence in (Z, p^t). As, (Z, p) is complete, thus (Z, p^t) is complete, so $\exists\ s, v \in Z$
$\lim_{m\to\infty} p^t(gs_m, s) = \lim_{m\to\infty} p^t(v_m, v) = 0$
By lemma, we get

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$$p(s,s)=\lim_{m\to\infty} p(gs_m,s)=\lim_{m\to\infty} p(gs_m,gs_m)$$
$$p(v,v)=\lim_{m\to\infty} p(gv_m,v)=\lim_{m\to\infty} p(gv_m,gv_m)$$

By condition and equation we get $\lim_{m\to\infty} p(gs_m,gs_m)=0$.
Thus follows as $p(u,u)=\lim_{m\to\infty} p(gs_m,u)=\lim_{m\to\infty} p(gs_m,gs_m)=0$, similarly $p(v,v)=\lim_{m\to\infty} p(gv_m,v)=\lim_{m\to\infty} p(gv_m,gv_m)=0$
We now prove that$T(s,v)=s, T(v,s)=v$.

Case1: As Z is a complete, $\exists\, s,v\in Z$
$\lim_{m\to\infty} s_m=s, \lim_{m\to\infty} v=v$ we prove that (s,v) is coupled coincidence point of T and g.

$$\begin{aligned}
s&=\lim_{m\to\infty} gs_{m+1}=\lim_{n\to\infty} T(s_m,v_m)=T(\lim_{m\to\infty} s_m,\lim_{m\to\infty} v_m)\\
v&=\lim_{m\to\infty} gv_{m+1}=\lim_{m\to\infty} T(v_m,s_m)=T(\lim_{m\to\infty} v_m,\lim_{m\to\infty} s_m)
\end{aligned}\tag{2.13}$$

As g is continuous, we attain

$$\lim_{m\to\infty} g(gs_m)=gs,\ \lim_{m\to\infty} g(gv_m)=gv. \tag{2.14}$$

Commutativity of T and g gives

$$\begin{aligned}
g(gs_{m+1})&=g(T(s_m,v_m))=T(gs_m,gv_m)\\
g(gv_{m+1})&=g(T(v_m,s_m))=T(gv_m,gs_m).
\end{aligned}\tag{2.15}$$

By continuity of T, $\{g(gs_{m+1})\}$ is converges to $T(s,v)$ and $\{g(gv_{m+1})\}$ converges to $T(v,z)$. From uniqueness of the limit and (2.14), we accomplish $T(s,v)=gs$ and $T(v,s)=gv$, consequently, T and g possesses a coupled incident point.
Case2: Presuppose that the condition (a) and (b) of the result holds.
The sequence $\{gs_m\}\to s$, $\{gv_m\}\to v$

$$\begin{aligned}
p(T(s,v),gs)&\le p(T(s,v),gs_{m+1})+p(gs_{m+1},gs)\\
&=p(T(s,v),T(s_m,v_m))+p(gs_{m+1},gs)\\
&\le p(gs,gs_m)-\psi(p(gv,gv_m))\\
&+L\,min\{p(gs,T(s,v)),p(gs_m,T(s_m,v_m)),p(gs_m,T(s,v)),p(gs,T(s_m,v_m))\}+p(gs_{m+1},gs)
\end{aligned}$$

Letting $m\to\infty$, we have $p(T(s,v),s)\le 0$
Thus $T(s,v)=s$, correspondingly, in similar way we can prove that $T(v,s)=v$. □

Theorem 2.2. *Presuppose the assumptions of Theorem 3.1 hold. Presuppose there exists $z\in Z$ which is comparable to s and v for every $s,v\in Z$. Thus T and g possesses only one coupled coincidence point.*

Proof. Succeeding the proof of Theorem 3.1, the arrangement of coupled coincidence points of T and g is non-empty. We will prove that coupled coincidence

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points are (s, v) and $(\acute{s}, \acute{v})$, then

$$\begin{aligned} g(s) &= T(s, v),\ g(v) = T(v, s) \\ \text{and } g(\acute{s}) &= T(\acute{s}, \acute{v}),\ g(\acute{v}) = T(\acute{v}, \acute{s}), \end{aligned}$$

then

$$gs = g\acute{s} \text{ and } gv = g\acute{v}. \tag{2.16}$$

Select $(d, z) \in Z \times Z$ comparable with both.

Let $d_0 = d, z_0 = z$ and choose $d_1, z_1 \in Z$ so that $gd_1 = T(d_0, z_0)$ and $gz_1 = T(z_0, d_0)$.
Then, similarly to the evidence of Theorem 3.1, we can inductively define sequences $\{gd_m\}$ and $\{gz_m\}$ as follows

$$gd_{m+1} = T(d_m, z_m) \ \ and \ \ gz_{m+1} = T(z_m, d_m).$$

Since $(gs, gv) = (T(s, v), T(v, s))$ and $(T(d, z), T(z, d)) = (gd_1, gz_1)$ are comparable, then $gs \leq gd_1$ and $gv \geq gz_1$. It is easy to prove using the mathematical induction,

$$gs \leq gd_m \ \ gv \geq gz_m \ \ \forall\, m \in \mathbb{N}.$$

Now, from the contractive condition (1)

$$\begin{aligned} p(gs, gs_{m+1}) &= p(T(s, v), T(s_m, v_m)) \\ &\leq p(gs, gs_m) - \psi(p(gv, gv_m)) \\ &\quad + L\, min\{p(gs, T(s, v)), p(gs_m, T(s_m, v_m)), p(gs_m, T(s, v)), p(gs, T(s_m, v_m))\} \\ &\leq p(gs, gs_m) - \psi(p(gv, gv_m)) \end{aligned} \tag{2.17}$$

Similarly

$$p(gv, gv_{m+1}) = p(gv, gv_m) - \psi(p(gs, gs_m)) \tag{2.18}$$

Adding (2.17) and (2.18), we get

$$p(gs, gs_{m+1}) + p(gv, gv_{m+1}) \leq p(gs, gs_m) + p(gv, gv_m) - [\psi(p(gv, gv_m)) + \psi(p(gs, gs_m))] \tag{2.19}$$

This implies

$$p(gs, gs_{m+1}) + p(gv, gv_{m+1}) \leq p(gs, gs_m) + p(gv, gv_m) \tag{2.20}$$

Thus, the sequence is non-increasing. hence, there exist $\alpha \geq 0$.

$$\lim_{m \to \infty} p(gs, gs_m) + p(gv, gv_m) = \alpha \tag{2.21}$$

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We shall prove that $\alpha = 0$. Presuppose in contrary, $\alpha > 0$. Take $m \to \infty$ in equation (2.21), we have

$$\alpha \leq \alpha - \psi(\alpha) < \alpha \tag{2.22}$$

a contradiction. Therefore, $\alpha = 0$, that is

$$\lim_{m\to\infty} p(gs, gs_m) + p(gv, gv_m) = 0.$$

It implies

$$\lim_{m\to\infty} p(gs, gs_m) = \lim_{m\to\infty} p(gv, gv_m) = 0.$$

Similarly, we can prove

$$\lim_{m\to\infty} p(g\acute{s}, gs_m) = \lim_{m\to\infty} p(g\acute{v}, gv_m) = 0.$$

From last equalities, we have $gs = g\acute{s}$ and $gv = g\acute{v}$. □

Example 2.3. Presume $Z = [0, 1]$ with usual partial metric p defined as $p : Z \times Z \to$ [0,1] with $p(s, v) = max(s, v)$. The (Z, p) is complete partial metric space for any $s, v \in Z$.

$$p(s, v) = |s - v|$$

Thus (Z, p^t) is complete Euclidean metric space.
Presume the mapping $T : Z \times Z \to Z$ given as $T(s, v) = \frac{2s-v}{4}; s \geq v$

Take $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(t) = \frac{t}{4}$

As, T has the property of mixed g-monotone property and is continuous.
Now, we discuss the following possibilities for (s, v) and (u, z) with $gs \leq gu,\ gv \geq gz$

Case 1- If $(s, v) = (u, z) = (0, 0)$
Then clearly $p(T(s, v), T(u, z)) = 0$
Thus (1) holds.
Case 2- If $(s, v) = (u, z) = (1, 0)$
Then LHS of (1)
$= p(T(s, v), T(u, z)) = p(T(1, 0), T(1, 0)) = p(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$,
which is less than RHS of (1)
Thus (1) holds.
Case 3- If $(s, v) = (u, z) = (1, 1)$
Thus (1) holds.
Case 4- If $(s, v) = (1, 0); (u, z) = (0, 0)$
Thus (1) holds.
Case5- If $(s, v) = (1, 0); (u, z) = (1, 1)$
Thus (1) holds.

Therefore, all the properties of Theorem 3.1 are gratified.
Also, g and T possesses unique coupled coincidence point as $(0, 0)$.

Conclusions

As introduced toward the start of this work, Bhaskar and Laksmikantham, stretch out this hypothesis to partially ordered metric spaces and present the idea of coupled fixed point for mixed-monotone map.

Acquiring results as concerns the presence and the uniqueness of certain coupled coincidence point hypotheses for a map possesses the property of mixed g-monotone in partial metric spaces.

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Numerical Study of Viscous Dissipation, Suction/Injection Effects and Dufour Number also with Chemical Reaction Impacts of MHD Casson Nanofluid in Convectively Heated Non-Linear Extending Surface

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Abstract

This numerical study looked at the effects of thermophoresis diffusion, Brownian motion parameter influences, and suction/injection influence in a hydromagnetic (MHD) Casson nanofluid in a convectively heated nonlinear extending surface (in 2D). Using similarity transformations, the leading partial differential equations (PDEs) are renewed into a set of ordinary differential equations (ODEs) with suitable boundary conditions, and then numerically resolved using a 4^{th}order Runge-Kutta approach based on the shooting technique and the MATLAB application. Graphs are used to investigate the effects of dimensionless parameters such as local Grashof temperature and concentration parameter, permeability, Joule impact, thermo radiative impression, Dufour and chemical reactive impression on nanoparticle volume fraction profiles, temperature, and movement. Tables and graphs are used to examine other characteristics of importance, such as the skin friction coefficient, heat, and mass transfer in a variety of situations, as well as the relationship between these parameters.

Key words: Casson Nanofluid; MHD; Heat Generation/Absorption, Thermophoresis Diffusion, RK-4^{th} order.

1 Introduction

There are several applications, including the learning of non-Newtonian fluids across an extended sheet, which was completed with extreme kindness. Although the elasticity of non-Newtonian fluid behaviour may be assessed, their fundamental equations are occasionally used to classify the rheological features. The fundamental relations in non-Newtonian fluids are extra difficult because they provide the rheological non-dimensional characteristics. Non-Newtonian fluids include a variation of fluids used in the oil industry, as well as cooling courses for micro-ships, unclosed-flow switching, and multiplex systems.

Because of its applications in paramedical sciences, geo and astrophysics, oil reservoirs, and geothermal engineering, free convective heat transport is a mean-

ingful part of fluid dynamics. The term "thermal radiation" refers to a method of converting internal energy into electromagnetic waves. Thermal radiation and nonlinear thermal radiation are employed in a variety of applications, including space vehicles, paper and glass manufacture, gas turbines, space technologies, and hypersonic combat. The flow model is based on a mix of Tiwari and Das models, as well as the Buongiorno's model.The influence of MHD Casson fluid flow through a convective surface with crossdiffusion, chemical reaction, and nonlinear radiative heat is accounted for using convective and boundary conditions, according to Ramudu et al. [1].Butt et al. [2] assessed the entropy generative impression of a flow traversed by a permeable stretched surface of hydromagnetic Casson nanofluid. Afify [3] addresses Casson nano fluid's work in the presence of viscid dissipative impression on a stretched sheet with slip limits. AlHossainy et al [4] address a SQLM (spectral quasi linearization method) mathematical work for the impact of stress on hydromagneto nanofluid flow with permeability influence in three dimensions. The timedependent nonlinearly convective stream of thin film nano liquid across an inclined stretchable sheet with a magnetic effect was studied by Saeed et al. [5]. The use of this current fractional operator to investigate Newtonian heating impacts for the generalized Casson fluid flow is the focus of Tassaddiq et al. [6] research. In this study, the MHD and porous impacts of such fluids are also taken into account. MHD Casson nanofluid (Ag and Cu water) boundary layer flow and heat transference across a stretched surface through a porous mode were studied by Siddiqui and Shankar [7]. Faraz [8] investigated a mathematical study on an axisymmetric Casson nanofluid flow over a radially stretched sheet with hydromagnetic impact. Hady et al. [9] inspected the radiative effect and heat transmission of a viscid nanofluid across a nonlinear stretched sheet. In the company of porous mode, Mahantha and Shaw [10] proposed a 3dimensional convective Casson fluid flow with convective limits passing through a linear stretched sheet. Vendabai [11] investigate a hydromagnetic boundary layer Casson nanofluid flow passing through an upright exponentially stretched cylinder with transverse magneto impact and heat generating or absorptive impression. Alotaibi et al. [12] investigated the influence of viscid dissipative impact over a convectively intensive nonlinear spreading surface, as well as suction or injection and heat absorption or generation impacts, on a hydromagnetic boundary layer flow of Casson nanofluid flow. Oyelakin et al. [13] studied a flow of timedependent Casson nanofluid across a stretched surface with thermal radiative imprint and slip limiting settings.

Many studies of Newtonian and nonNewtonian fluids have been directed in order to examine the impacts of fluid movements, as well as various types of nanofluid flows across various surfaces. In fresh years, a large number of inspections on the boundary layer flow of Casson nanofluids in a variety of geometries have been carried out. Ullah et al. [14] looked at the effect of thermo radiative, convective limiting circumstances, and heat generation/absorption on a timedependent hydromagnetic mixed connective slip Casson fluid flow, as well as chemically reactive influence, on a nonlinearly stretched sheet in a porous mode. The local fractional linear transport equations (LFLTE) in fractal porous media are studied by Singh et al. [15]. Dwivedi and Singh [16] produced a new finite difference collocation approach that was designed using the Fibonacci polynomial and then used to one super and two sub-diffusion problems with better reliability. Imtiaz et al. [17] examined how a convective Casson nanofluid flow goes through a stretched cylinder and the restrictions that come with it. Eid and Mahny [18] describe a computational study to determine the heatgenerating influence of Sisko nanofluid across a nonlinear stretched sheet with porous mode. Eid [19] investigated a twophase nanofluid flow with hydromagnetic influence, as well as chemical reactive and heat generating effects, across an exponentially stretched sheet. Chemical reaction effects on a convectively heated nonlinear stretched surface of Carreau nanofluid were explored by Eid et al [20].

Eid [21] investigated the chemical reactive effects of H_2O-NPs (nanoparticles) in unsteady and stagnation point flow on a stretched sheet in the soaking porous mode. Mustafa and Khan [22] investigated a Casson model flow over a nonlinear stretched sheet with magneto impacts. Wahiduzzaman et al [23] did a mathematical investigation of hydromagnetic Casson fluid flow in the presence of porous mode passing through a nonisotherm stretched sheet. The Casson nanofluid flow between stretched discs with radiative influence was deliberated by Khan et al [24]. The viscid dissipative impression of hydromagnetic Casson nanofluid passing through permeable stretched sheet was observed by Besthapu and Bandari [25]. Pramanik [26] discusses the thermo radiative and Nusselt number impressions in the presence of a porous mode of nonNewtonian Casson flow passing over an exponentially stretched surface.

The vast range of commercial and manufacturing experiments of flow behaviour across stretched surfaces has attracted numerous writers, including artificial fibres, metallic sheet manufacture, petroleum industries, metal spinning, polymer processing, and so on. Reddy[27] examined the thermal radiative effect and chemically reactive impact of a hydromagnetic Casson fluid flow over an exponentially persuaded permeable stretched surface. Vijayaragavan [28] used a permeable stretched sheet to investigate the heat source or sink effect of timeindependent hydromagnetic Casson fluid flow. Haq et al. [29] examined timedependeny free convection slip flow of secondgrade fluid across an endless hot inclined plate. Lahmar et al. [30] examined the impacts of thermal conductivity and Nusselt number on the squeezing of a timedependent nanofluid by a tending magneto. Nadeem et al. [31] considered a nonNewtonian shear thinning Casson fluid flow with permeability impact passes across a stretchy linear sheet. Mass transference of hydromagnetic Casson fluid with suction and chemical reactive impression was explored by Shehzad et al [32]. Dahab et al. [33] investigated the influence of extending surface over a nonlinearly heated extending surface using hydromagnetic Casson nanofluid flow. Ibrahim et al. [34] conducted a mathematical investigation of a dissipative hydromagnetic mixed convective Casson nanofluid with chemical reactive effect across a nonlinear permeable stretched sheet with heat source impression. Hayat et al. [35] took into account mixed convective stagnation point Casson fluid flow as well as convective constraints. The goal of Puneeth et al. [36] is to figure out what function mixed convection, Brownian motion, and thermophoresis play in the dynamics of a Casson hybrid nanofluid in a bidirectional nonlinear stretching sheet. The heat transfer and entropy of an unstable Casson nanofluid flow, where fluid is positioned across a stretched flat surface flowing nonuniformly, were explored by Jamshed et al. [37]. Over a nonlinear stretched sheet, Shah et al [38] discussed chemically reactive hydromagnetic Casson nanofluid flow with radiation influence and entropy generating impression. Soret or diffusion thermo or thermo difusion effect is described as matter diffusion caused by a gradient of heat, whereas Dufour efect is defined as heat diffusion caused by a gradient of concentration. For excessively big temperature and concentration gradients, these consequences have played a substantial influence. The majority of the time, these two impacts are regarded as secondorder effects. Its uses include contaminant movement in groundwater, chemical reactors, and geosciences. Several academics are drawn to the field of heat flux mass transfer because of its wide range of applications in numerous fields. Fiber optics manufacturing, plastic emulsion, glass cutting, nanoelectronics freezing, catalytic reactors, wire drawing, and improved oil extraction are all examples of Brownian motion effects and thermophoresis in the scientific and technical sphere.

The current study attentions on the Casson nanofluid flow over a nonlinear inclined stretching surface with Buoyancy and Dufour impacts, as a result of the above mentioned literature review and the rising need for nonNewtonian nanofluid

flows in industry and engineering. When compared to Newtonian based nanofluid flow, Casson nanofluid is more useful for cooling and friction-reducing agents. The goal of this research is to show a comprehensive mathematical investigation of the impression of buoyancy force, permeability, joule heating impression, and chemical reactive with heat generative or absorption, suction or blowing impact, and viscid dissipative impression of 2-dimensional hydromagneto Casson fluid flow permits through nonlinear outspreading plate along with CBC (convective boundary conditions) ref [12]. PDEs (dimension form) of existing effort were turned into ODEs with the support of several similarity transformations. The RK4th order process cracked nondimensional ODEs with the help of the shooting procedure. MATLAB software is used to create graphs and tables that highlight the rooted parameter behaviour.

2 Problem Structure:

Deliberate hydromagnetic convective nanofluid flow in the section ($y > 0$) over an exponentially extensible sheet as 2-dimensional incompressible (density is constant) time-independent viscid along with the impact of the viscous dissipative and several non-dimensional parameters. X-axis indicated for surface and y-axis erect to surface. The extensible surface is projected to have a velocity outline of the power law $u_w(x) = ax^n$ where a, n is non-zero constants. Magnetic field is covered with $B(x) = B_0 x^{(n-1/2)}$ and the electric field is zero, but the induced magnetic field is unnoticed by the weak magnetic amount of Reynolds. $T_w(x) = T_\infty + Ax^n$, where A is non-zero secure rate, T_∞ displays the free stream temperature. C_∞ displays the ambient nano-particles concentration. Flow chart of existing work is exposed in [Figure 1].

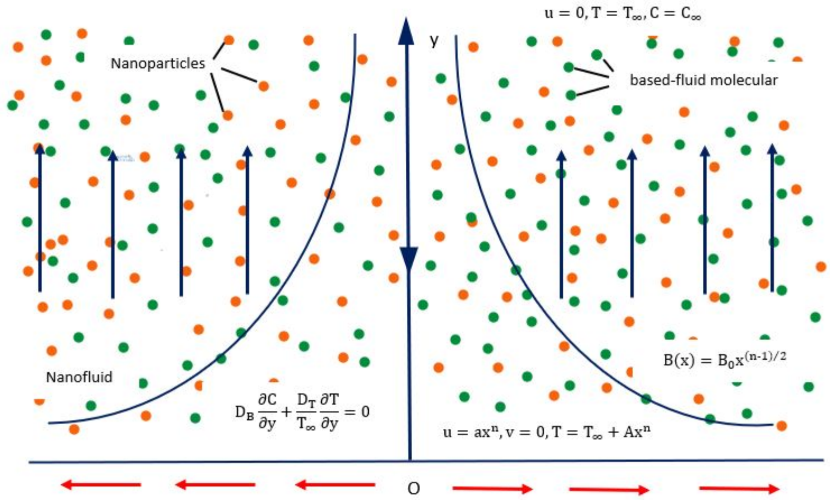

Figure 1: Bodily Modal of recent effort.

The Casson fluid rheological state equation [references [21], [31]] is:

$$\tau_{ij} = \begin{cases} 2(\mu_B + \frac{\tau_y}{\sqrt{2\pi}})e_{ij} & \text{if } \pi > \pi_c \\ 2(\mu_B + \frac{\tau_y}{\sqrt{2\pi}})e_{ij} & \text{if } \pi < \pi_c \ . \end{cases} \quad (1)$$

Where deformation component rate product $\pi = e_{ij}e_{ij}$, where e_{ij} displays the $(i,j)^{th}$ deformation component rate, π is the multiple of the sections of deformation

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component rate product, π_c indicates to the non-Newtonian fluid critical rate of this deformation component rate product, μ_B signifies Casson fluid plastic viscosity, τ_y shows yield stress.

Consider the apparatuses of velocity function $V = [u(x, y), v(x, y), 0]$, the temperature function $T = T(x, y)$ and concentration function $C = C(x, y)$. The motion, temperature and the concentration relations in a Casson nanofluid are inscribed as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \nu(1+\frac{1}{\beta})\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2(x)u}{\rho_f} - \frac{\nu}{k'}u + g_0\beta_T(T-T_\infty) + g_0\beta_C(C-C_\infty), \tag{3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{Q_0}{\rho c_p}(T-T_\infty) + \tau[D_B(\frac{\partial T}{\partial y}\frac{\partial C}{\partial y}) + \frac{D_T}{T\infty}(\frac{\partial T}{\partial y})^2] + \frac{\mu}{\rho c_p}(1+\frac{1}{\beta})(\frac{\partial u}{\partial y})^2$$
$$+\frac{\sigma B^2(x)u^2}{\rho c_p} - \frac{1}{\rho c_p}\frac{\partial q_r}{\partial y} + \frac{D_B K_T}{c_s c_p}\frac{\partial^2 C}{\partial y^2}, \tag{4}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2} - k_1(C - C_\infty). \tag{5}$$

where, u and v show the x-axis and y-axis velocity apparatuses respectively, ν displays the kinematic viscosity, $\beta = \mu_B\frac{\sqrt{2\pi_C}}{\tau_y}$ indicates the Casson fluid parameter, σ displays the conductivity electrical field, ρ indicates the fluid density, the thermophoresis and Brownian diffusions coefficients are direct by D_T, and D_B respectively, Q_0 specifies the dimensional heat source or sink coefficient, α shows the thermal diffusivity, T pointed for temperature,$\tau = \frac{(\rho C)_p}{(\rho C)_f}$ directs the ratio of the effective heat capacity to efficient liquid heat capacity, and C_p indicates the specific heat.

The existing work boundary conditions are specified by

$$u = u_w = ax^n,\ v = v_w,\ T = T_w = T_\infty + Ax^n,\ D_B\frac{\partial C}{\partial y} + \frac{D_T}{T_\infty}\frac{\partial T}{\partial y} = 0,\ \ at\ y = 0.$$
$$u \to 0, \qquad T \to T_\infty, \qquad C \to C_\infty, \qquad at\ \ y \to \infty. \tag{6}$$

Where, $a > 0$ is for the stretching channel walls.

The Roseland approximation of the radiative heat flux is arranged by

$$q_r = \frac{-4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y}, \tag{7}$$

Here T^4 as a linear relation of temperature via Taylor's sequence expansion about T_∞ and ignoring advanced terms, thus

$$T^4 \approx 4{T_\infty}^3 T - {T_\infty}^4. \tag{8}$$

In view of the similarity transformation

$$u = ax^n f'(\eta), \qquad v = -ax^{(n-1)/2}\sqrt{\frac{\nu}{a}}(\frac{n+1}{2}f(\eta) + \frac{n-1}{2}\eta f'(\eta)), \tag{9}$$

$$\eta = \sqrt{\frac{a}{\nu}}x^{(n-1)/2}y, \qquad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \qquad \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}. \tag{10}$$

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With (7)-(10), equations (2)-(6)are reduced to the next arrangement.

$$(1+\frac{1}{\beta})f''' - n(f')^2 + (\frac{n+1}{2})ff'' - Mnf' + k_2 f' + G_T\theta + G_C\phi = 0, \tag{11}$$

$$((1+Nr)/Pr)\theta'' - nf'\theta + (\frac{n+1}{2})f\theta' + Nb\theta'\phi' + Nt(\theta')^2 + (1+\frac{1}{\beta})Ec(f'')^2$$
$$+Q\theta + MnEc(f')^2 + Du\phi'' = 0, \tag{12}$$

$$\phi'' + (\frac{n+1}{2})Scf\phi' + \frac{Nt}{Nb}\theta'' - ScK\phi = 0. \tag{13}$$

With limit circumstances

$$f(0) = f_w, \quad f'(0) = 1, \quad \theta(0) = 1, \quad Nb\phi'(0) + Nt\theta'(0) = 0, \quad at \;\; \eta = 0. \tag{14a}$$
$$f'(\infty) \to 0, \quad \theta(\infty) \to 0, \quad \phi(\infty) \to 0, \quad at \;\; \eta \to \infty. \tag{14b}$$

Where, $Mn = \frac{\sigma B^2(x)}{\rho a x^{n-1}}$ shows the magnetic parameter, $k_2 = \frac{\nu}{k' a x^{n-1}}$ indicates the permeability parameter, $G_T = \frac{g_0\beta_T(T_w - T_\infty)}{a^2 x^{2n-1}}$ and $G_C = \frac{g_0\beta_C(C_w - C_\infty)}{a^2 x^{2n-1}}$ indicate the local temperature and concentration Grashof number respectively, $Pr = \frac{\nu}{\alpha}$ shows the Prandtl number, $Q = \frac{Q_0}{\rho a c_p x^{n-1}}$ indicates the heat generation or absorption, $Nr = \frac{16\sigma^* {T_\infty}^3}{3kk^*}$ directs the radiation parameter, $Ec = \frac{u_w^2}{c_p(T_w - T_\infty)}$ shows the Eckert number, $Nb = \frac{\tau D_B(C_w - C_\infty)}{\nu}$ directs the Brownian motion parameter, $Nt = \frac{\tau D_T(T_w - T_\infty)}{\nu T_\infty}$ indicates the thermophoresis diffusion influence, $Du = \frac{D_B K_T(C_w - C_\infty)}{c_s c_p \nu(T_w - T_\infty)}$ specifies the Dufour number, $Sc = \frac{\nu}{D_B}$ directs the Schmidt number, $K = \frac{k_1}{a x^{n-1}}$ shows the chemical reaction parameter, and $f_w = \frac{-2v_w}{(n+1)\sqrt{\nu a x^{n-1}}}$ indicates the suction or blowing parameter.

Physical Quantities:

Skin Friction Coefficient(C_{fx})**:** The skin friction coefficient is known as follows:

$$C_{fx} = \frac{\tau_w}{\rho u_w^2}, \;\; where \;\; \tau_w = \mu_B(1+\frac{1}{\beta})(\frac{\partial u}{\partial y})_{y=0}. \tag{15}$$

Heat Transfer Coefficient: The non-dimensional Nusselt number (Nu_x) is given by

$$Nu_x = \frac{xq_w}{k(T_w - T_\infty)}, \;\; where \;\; q_w = -k(\frac{\partial T}{\partial y})_{y=0}. \tag{16}$$

Mass Transfer Coefficient: The rate of mass transfer is derived by a Sherwood number (Sh_x) which is given by

$$Sh_x = \frac{xm_w}{D_B(C_w - C_\infty)}, \;\; where \;\; m_w = -D_B(\frac{\partial C}{\partial y})_{y=0}. \tag{17}$$

After solving the equation (15), (16) and (17) with equation (9) and (10), we gain

drag force

$${Re_x}^{1/2}C_{fx} = (1+\frac{1}{\beta})f''(0),$$

local Nusselt

$${Re_x}^{1/2}Nu_x = -\theta'(0),$$

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local Sherwood

$$Re_x{}^{1/2}Sh_x = -\phi'(0). \tag{18}$$

Where,τ_w indicates the wall shear stress, k signifies the thermo nano-fluid conductivity, q_w shows the surface heat flux, and m_w directs the surface mass flux, $Re_x = \frac{u_w x}{\nu}$ shows the local Reynolds number.

We explain the reduces equations (11)-(13) with limitations (14a) and (14b) using Runge -Kutta fourth-order method along with shooting technique.

3 Results and Discussion:

The impacts of the numerous types of non-dimensional parameters values to have a physical considerate of the work like, the magneto impact Mn, f_w (suction or blowing), β Casson parameter, heat source or sink impact Q, Dufour impact Du, Brownian diffusivity Nb, Eckert parameter Ec, thermophoresis diffusivity Nt, radiative impact Nr, Prandtl effect Pr, Schmidt impact Sc, chemically reactive influence K, permeability influence k_2, Grashof number impact (G_T and G_C), and power law index n, over the momentum graphs $f'(\eta)$, temperature graphs $\theta(\eta)$, and concentration graphs $\phi(\eta)$ discussed through graphs [Figure 2 - Figure 23] and tables along with the influence of drag force $Re_x{}^{1/2}C_{fx}$, local Nusselt $Re_x{}^{1/2}Nu_x$ local Sherwood $Re_x{}^{1/2}Sh_x$. Non-dimensional equations (11)-(13) with limitations (14a) and (14b) solved by Runge- Kutta fourth-order method with shooting technique. After we find the values of heat transfer and mass transfer and draw the graph by MATLAB software. Consider the values of parameters $n = 3$ or $1, \beta = 0.1$ or $1, Mn = 0.1, k_2 = 0.1, Pr = 0.1, Q = 0.1, Nr = 0.1, Sc = 0.1, K = 0.1, G_T = 0.1, G_C = 0.1, Ec = 0.1, Du = 0.1, Nb = 0.1, Nr = 0.1, f_w = 0.1$.

The influence of the Casson nanofluid parameterβ on the momentum profile with power law index $n = 3$ and 1 is seen in [Figure 2] . The momentum graph drops as β increases in both situations of n, owing to the upsurge in plastic dynamic viscosity, which produces hindrance in the fluid flow. When $n = 3$, the momentum of the Casson nanofluid is greater than that of the Casson nanofluid when $n = 1$. With the track erect to the x-axis, [Figure 3] displays the lowering momentum impact of increased magnetic number Mn owing to the solid Lorentz force, which generates greater resistance in the fluid flow in both situations of Casson fluid $\beta = 0.1$ and 1. [Figure 4] depicts the failure velocity with suction or blowing parameter f_w due to nanofluid heat and thermolayer thickness for $n = 3$ and $n = 1$. When the permeability parameter k_2 rises to $\beta = 0.1$ and 1, the momentum decreases as seen in [Figure 5]. The momentum upsurges due to a rise in buoyant force with growing values of local concentration Grashof number G_C and local temperature Grashof number G_T, in [Figure 6] and [Figure 7] for $\beta = 0.1$ and 1, respectively.

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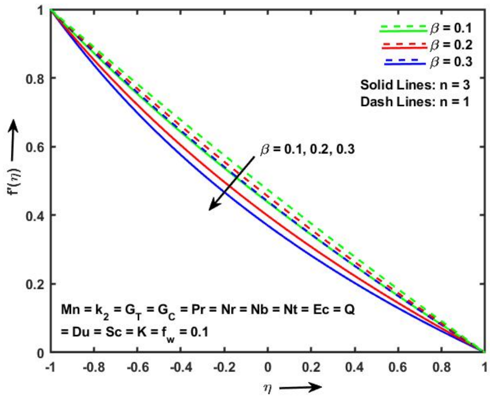

Figure 2: Velocity display of Casson fluid parameter β.

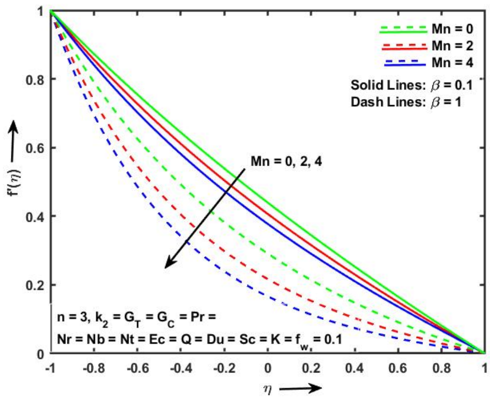

Figure 3: Velocity display of Magnetic parameter Mn.

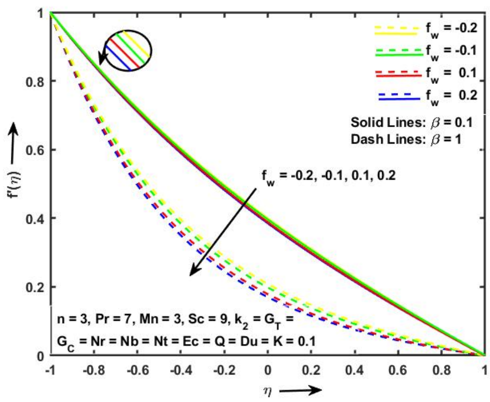

Figure 4: Velocity display of parameter f_w.

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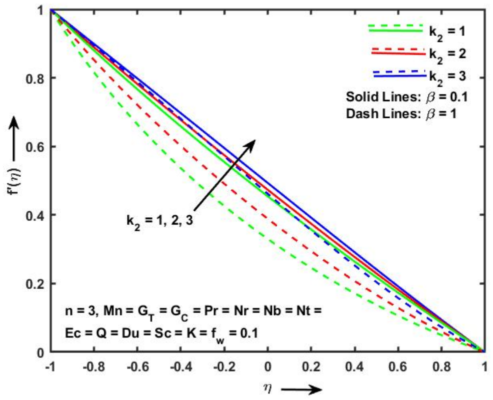

Figure 5: Velocity display of parameter k_2.

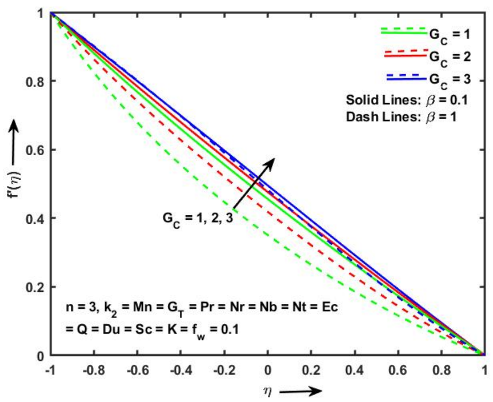

Figure 6: Velocity display of parameter G_C.

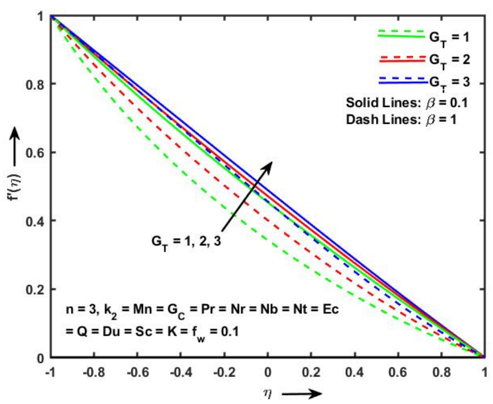

Figure 7: Velocity display of parameter G_T.

[Figure 8] illustrations the result of increasing the Dufour number on temperature between $\beta = 0.1$ and 1. [Figure 9] depicts the increasing influence of temperature on Eckert number Ec. The temperature rise as a result of the viscous dissipative term, which generates heat as a result of frictional heating between the fluid constituents. This additional heat resulted in a rise in temperature, which was connected to an increase in boundary layer breadth. [Figure 10] shows how temperature rises when the radiation parameter Nr between $\beta = 0.1$ and 1 increases. The $k*$ (absorption coefficient) lowers when there is an ascendant in Nr. An increase in f_w falloffs the nanofluid heat and thermolayer width owing to heated nanofluid pulled close to sheet is seen in [Figure 11]. As a result, in both scenarios of $\beta = 0.1$ and 1 with increase f_w, the temperature decreases. The Prandtl number Pr represents the ratio of momentum and thermal diffusivity. The temperature is lowered when Pr is increased (in [Figure 12]). Due to thermal layer thickness, [Figure 13] depicts the temperature increase with increasing heat production or absorption parameter Q (heat create in fluid for $Q > 0$ and heat absolve in fluid for $Q < 0$).

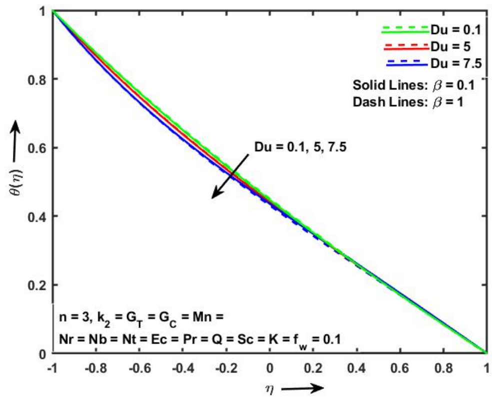

Figure 8: Temperature display of parameter Du,

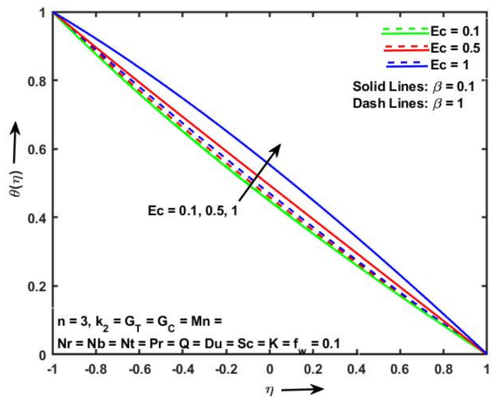

Figure 9: Temperature display of parameter Ec.

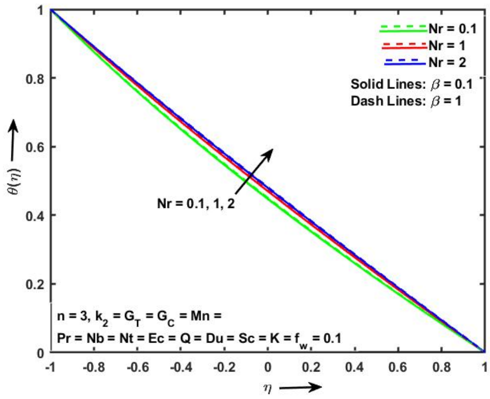

Figure 10: Temperature display of parameter Nr.

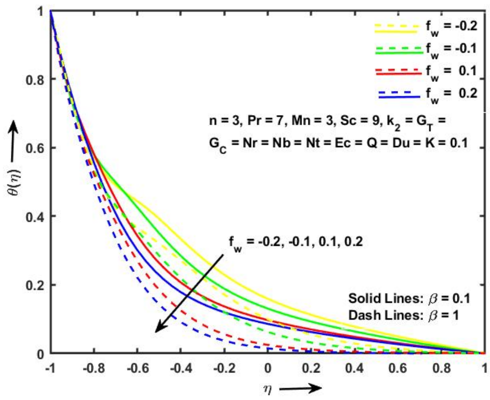

Figure 11: Temperature display of parameter f_w.

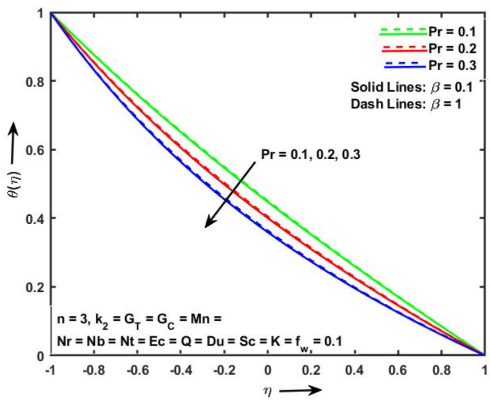

Figure 12: Temperature display of parameter Pr.

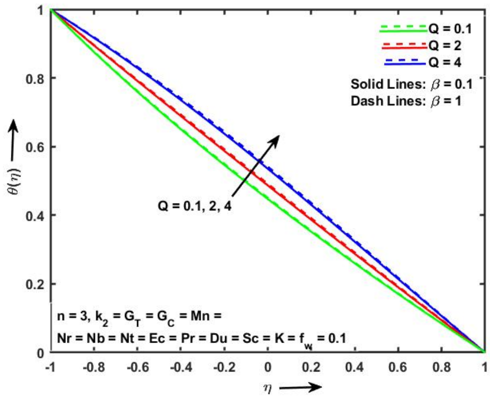

Figure 13: Temperature display of parameter Q.

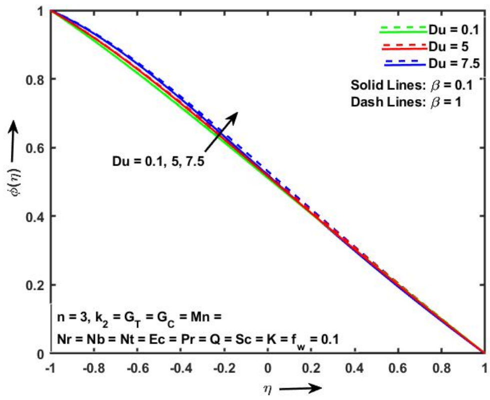

Figure 14: Concentration display of parameter Du.

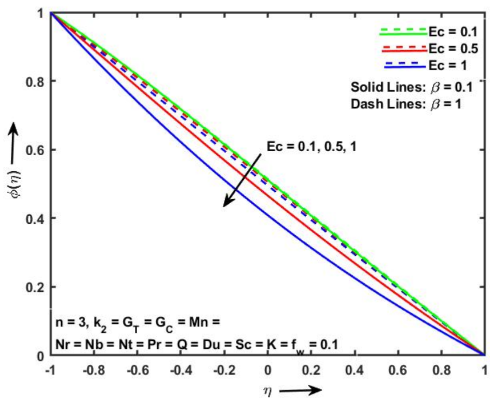

Figure 15: Concentration display of parameter Ec.

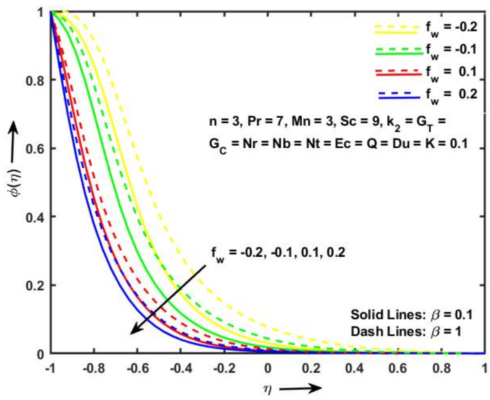

Figure 16: Concentration display of parameter f_w.

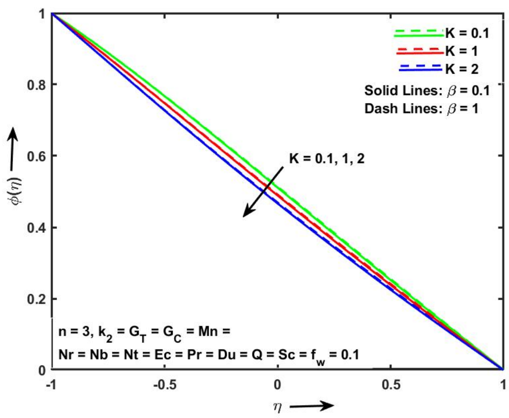

Figure 17: Concentration display of parameter K.

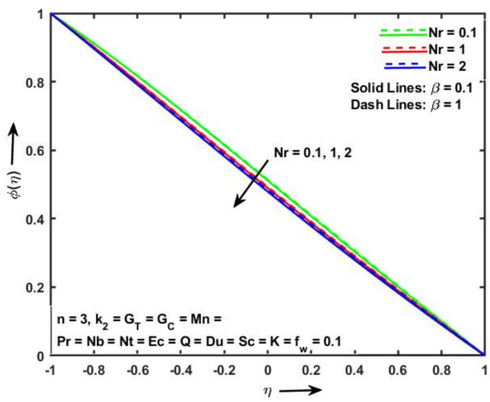

Figure 18: Concentration display of parameter Nr.

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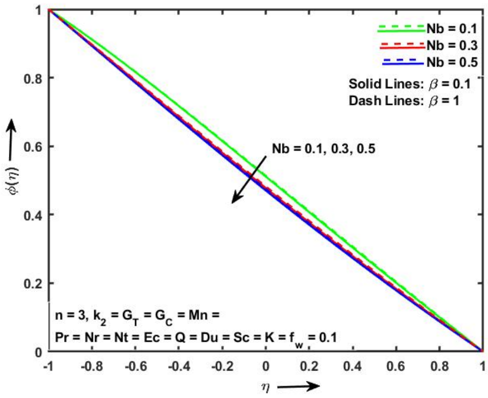

Figure 19: Concentration display of parameter Nb.

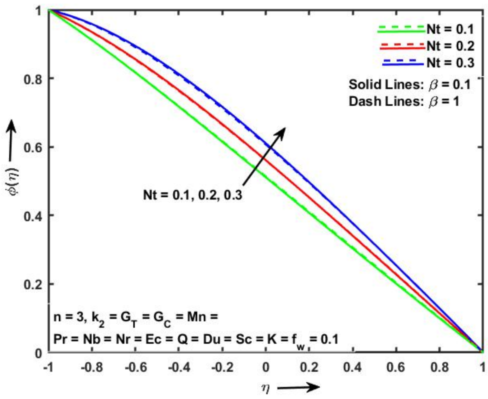

Figure 20: Concentration display of parameter Nt.

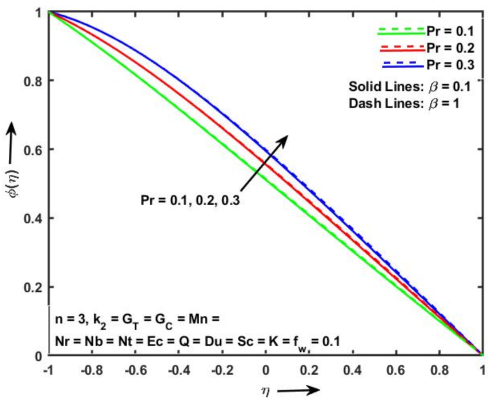

Figure 21: Concentration display of parameter Pr.

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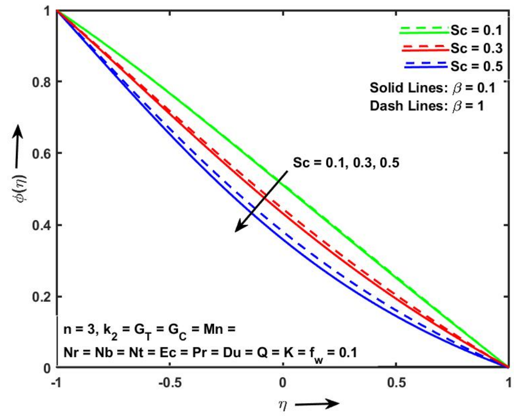

Figure 22: Concentration display of parameter Sc.

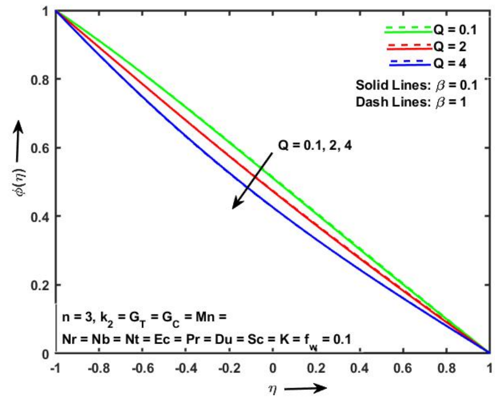

Figure 23: Concentration display of parameter Q.

[Figure 14] depicts an increment of concentration shape with growing Dufour number Du. [Figure 15]-[Figure 19] show the decay volume fraction graph for both $\beta = 0.1$ and 1 of Eckert parameter Ec, suction or blowing impact f_w, chemically reactive imapct K, and radiation parameter Nr. [Figure 20] describes the falling impact of concentration profile of upsurge Brownian parameter Nb for both $\beta = 0.1$ and 1. Similarly, impact of decay volume fraction of increase thermo-diffusion influence showing in [Figure 20]. [Figure 21] shows the effect of rise Prandtl number Pr on growing concentration profile for both $\beta = 0.1$ and 1. Increasing Schmidt number Sc and heat generation and absorption parameter Q impact with declines volume fraction influence explains in [Figure 22] and [Figure 23] respectively.

Table I: Numerical result of drag force coefficient, Nusselt number Nu, and Sherwood number of numerous parameters when $\beta = 0.1$ and 1 and $n = 3$.

Mn	k_2	G_T	G_C	f_w	Pr	Nr	Nb	Nt	Ec	Q	Du	Sc	K	$f''(0)$	$\theta'(0)$	$\phi'(0)$
0 **2** **4**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**-0.6589** **-0.7599** **-0.8538** **-1.1769** **-1.5325** **-1.8312**	**-0.6429** **-0.6269** **-0.6119** **-0.6389** **-0.6179** **-0.6009**	**-0.4222** **-0.4360** **-0.4492** **-0.4182** **-0.4352** **-0.4492**
0.1	**1** **2** **3**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**-0.6159** **-0.5609** **-0.5309** **-1.0129** **-0.7849** **-0.5249**	**-0.6459** **-0.6499** **-0.6549** **-0.6469** **-0.6579** **-0.6719**	**-0.4192** **-0.4162** **-0.4122** **-0.4122** **-0.4032** **-0.3942**
0.1	**0.1**	**1** **2** **3**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**-0.6169** **-0.5639** **-0.5119** **-0.9879** **-0.7638** **-0.5458**	**-0.6459** **-0.6499** **-0.6539** **-0.6489** **-0.6599** **-0.6699**	**-0.4192** **-0.4162** **-0.4132** **-0.4102** **-0.4022** **-0.3952**
0.1	**0.1**	**0.1**	**1** **2** **3**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**-0.6129** **-0.5559** **-0.4999** **-0.9723** **-0.7287** **-0.4903**	**-0.6459** **-0.6509** **-0.6549** **-0.6499** **-0.6629** **-0.6739**	**-0.4192** **-0.4152** **-0.4122** **-0.4092** **-0.4002** **-0.3922**
3	**0.1**	**0.1**	**0.1**	**-0.2** **-0.1** **0.1** **0.2**	**7**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**9**	**0.1**	**-0.785275** **-0.794275** **-0.8139** **-0.8229** **-1.5580** **-1.6079** **-1.7109** **-1.7639**	**-2.792727** **-3.240827** **-3.4749** **-3.1099** **-2.90922** **-3.4316** **-3.9492** **-3.7849**	**-0.107859** **-0.389959** **-2.2192** **-3.8882** **0.11995** **-0.0675** **-1.5985** **-3.0582**

0.1	0.1	0.1	0.1	0.1	0.1 0.2 0.3	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	-0.6639 -0.6639 -0.6639 -1.1969 -1.1969 -1.1969	-0.6419 -0.7779 -0.9079 -0.6379 -0.7729 -0.9039	-0.4222 -0.2868 -0.1568 -0.4188 -0.2838 -0.1518
0.1	0.1	0.1	0.1	0.1	0.1	0.1 1 2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	-0.6639 -0.6639 -0.6639 -1.1969 -1.1969 -1.1969	-0.6419 -0.5789 -0.5529 -0.6379 -0.5765 -0.5505	-0.4222 -0.4852 -0.5122 -0.4188 -0.4808 -0.5058
0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1 0.3 0.5	0.1	0.1	0.1	0.1	0.1	0.1	-0.6639 -0.6639 -0.6639 -1.1971 -1.1977 -1.1977	-0.6419 -0.6479 -0.6539 -0.6379 -0.6439 -0.6499	-0.4222 -0.5152 -0.5342 -0.4188 -0.5092 -0.5272
0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1 0.2 0.3	0.1	0.1	0.1	0.1	0.1	-0.6639 -0.6639 -0.6635 -1.1971 -1.1955 -1.1945	-0.6419 -0.6409 -0.6399 -0.6379 -0.6369 -0.6359	-0.4222 -0.2822 -0.1442 -0.4188 -0.2828 -0.1482
0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1 0.5 1	0.1	0.1	0.1	0.1	-0.6639 -0.6636 -0.6636 -1.1971 -1.1971 -1.1971	-0.6419 -0.5239 -0.3749 -0.6379 -0.6009 -0.5549	-0.4222 -0.5412 -0.6902 -0.4188 -0.4558 -0.5018
0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1 2 4	0.1	0.1	0.1	-0.6639 -0.6639 -0.6639 -1.1971 -1.1971 -1.1971	-0.6419 -0.5309 -0.4019 -0.6379 -0.5249 -0.3939	-0.4222 -0.5337 -0.6627 -0.4188 -0.5318 -0.6638
0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1 5 7.5	0.1	0.1	-0.6639 -0.6639 -0.6639 -1.1971 -1.1971 -1.1971	-0.6419 -0.7019 -0.7899 -0.6379 -0.7019 -0.7979	-0.4222 -0.3632 -0.2762 -0.4188 -0.3558 -0.2602

0.1	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1** **0.3** **0.5**	**0.1**	**-0.6639** **-0.6636** **-0.6642** **-1.1971** **-1.1979** **-1.1999**	**-0.6419** **-0.6409** **-0.6389** **-0.6379** **-0.6369** **-0.6349**	**-0.4222** **-0.5572** **-0.6972** **-0.4188** **-0.5388** **-0.6628**
0.1	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1** **1** **2**	**-0.6639** **-0.6639** **-0.6639** **-1.1971** **-1.1971** **-1.1981**	**-0.6419** **-0.6409** **-0.6409** **-0.6379** **-0.6369** **-0.6369**	**-0.4222** **-0.4832** **-0.5472** **-0.4188** **-0.4798** **-0.5438**

Table II: Mathematical outcome of drag force coefficient, Nusselt number Nu, and Sherwood number of Casson nanofluid parameter β when $n = 3$.

β	Mn	k_2	G_T	G_C	f_w	Pr	Nr	Nb	Nt	Ec	Q	Du	Sc	K	$f''(0)$	$\theta'(0)$	$\phi'(0)$
0.1 **0.2** **0.3**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**0.1**	**-0.5589** **-0.6049** **-0.6419** **-0.6639** **-0.7819** **-0.8732**	**-0.5389** **-0.5489** **-0.5519** **-0.6419** **-0.6469** **-0.6469**	**-0.4982** **-0.4872** **-0.4842** **-0.4222** **-0.4152** **-0.4142**

4 Conclusion:

The impact of hydromagnetic Casson nanofluid's boundary layer on a nonlinear stretching sheet in 2D with the impact of viscid dissipative impact, Dufour number Du, heat absorption or generation Q impact, and suction or blowing impact, among other things, is examined mathematically. The skinfriction coefficient grows as permeability and Grashof number increase. As the Dufour number improves, the Nusselt number drops, whereas the Sherwood number drops as the chemical reactive impression improves. The Runge Kutta 4^{th} order procedure, as well as the shooting technique and MATLAB software, are used to arrive at the mathematical answer. The following are some of the study's key findings:

- Momentum display decreases when β, Mn, and f_w rise, but increases as k_2, G_T, and G_C rise.
- The temperature graph decreases when Du, f_w, and Pr increase, and increases as Ec, Nr, and Q increase.
- The volume fraction distribution improved when Du, Nt, Pr increased and falloffs in Ec, f_w, K, Nr, Nb, Sc, and Q decreased.
- The skin friction coefficient decreases when Mn, Nr, f_w, Nb, Ec, Q, Sc, K increases, while it climbs as k_2, G_T, G_C, β, Pr, Nt, and Du increase.
- The Nusselt number increases as Mn, Nr, Nt, Ec, Q, Sc, and K levels rise, but decreases as k_2, G_T, G_C, β, f_w, Pr, Nb, and Du levels rise.
- The Sherwood number increased when k_2, G_T, G_C, and Nt increased, but decreased as Mn, β, f_w, Nb, Sc, and K increased.

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Non-polynomial spline solution of one dimensional singularly perturbed parabolic equations

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Abstract

In this paper, two-parameter singularly perturbed parabolic equations are examined by two level method using non-polynomial spline. We have used non-polynomial quadratic spline in space and finite difference discretization in time. Stability analysis is carried out. The approximate solution is shown to converge point-wise to the true solution. Numerical solution of singularly perturbed parabolic equations consisting of linear as well as non-linear has been solved. Three numerical examples are presented to show the efficiency and effectiveness of the developed method.

Key words: Two-parameter singularly perturbed problems; Non-polynomial splines; Stability analysis

Mathematics Subject Classification(2010): 65L10; 65D07

1

1 Introduction

We consider the two-parameter singularly perturbed one dimensional parabolic partial differential equation(PDE) of the form:

$$z_\kappa - \epsilon_d z_{\lambda\lambda} + \epsilon_c r(\lambda) z_\lambda + s(\lambda) z \quad = \quad g(\lambda, \kappa), \qquad (\lambda, \kappa) \in Q_T, \tag{1.1}$$

subject to

$$z \quad = \quad 0, \ (\lambda, \kappa) \in \partial S \times I, \tag{1.2}$$

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and

$$z(\lambda,0) \quad = \quad z_0(\lambda),\ \lambda \in S, \tag{1.3}$$

where, $Q_T = S \times I$, $S \equiv r : l < r < m$, $\partial S \equiv \{l\} \cup \{m\}$, $I \equiv (0,T)$ and $l(>0), m(>0) \in \mathbb{R}$, $z = z(\lambda,\kappa)$, $r(\lambda)$ and $s(\lambda)$ are continously differentiable functions and $g(\lambda,\kappa)$ is continuous function defined on Q_T. Also $0 < \epsilon_c \ll 1$ and $0 < \epsilon_d \ll 1$. The above problems occur in various fields of sciences, such as, elasticity, mechanics, chemical reactor theory and convection-diffusion process. There are numerous asymptotic expansion methods available for solution of problems of the above type. But there were difficulties in applying these asymptotic expansions in the inner and outer regions. Many researchers have derived numerical methods for solving singularly perturbed boundary value problems(SPBVPs). Scheme based on parametric spline functions has been developed by Khan et al.[5]. Fractional Kersten-Krasil'shchik coupled KdV mKdV System arising in multi-component plasmas have been numerically solved by Goswami et al.[3]. A uniform convergent numerical method is given by Clavero et al.[2] and Kadalbajoo et al.[4] to solve the one-dimensional time-dependent convection-diffusion problem. Sharma and Kaushik [8] solved a singularly perturbed time delayed convection diffusion problem on a domain which is rectangular. Zahra et al.[10], Aziz and Khan [1] have also used spline methods for solution of SPBVPs. An efficient numerical approach for fractional multi-dimensional diffusion equations with exponential memory is given by Singh et al.[9]. In recent past, Mohanty et al. [7] have solved singularly perturbed parabolic equations using methods based on spline in tension. In this paper, we develop a new algorithm for solving SPBVPs associated with homogeneous Dirichlet boundary conditions.

This paper is divided into 5 sections as follows: In Section 2, the non-polynomial spline scheme is derived. In Section 3, we discuss application of the method for SPBVPs with scheme of $O(k+h^2)$. Truncation error is also discussed in Section 3. In Section 4, stability analysis is carried out. In Section 5 three problems are solved which confirm theoretical behaviour along with the rate of convergence.

2 Non-polynomial Spline

We divide the *[l,m]* interval uniformaly as

$$l = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{n-1} < \lambda_n = m,$$

where

$$\lambda_i = l + ih, \quad 0 \le i \le n \quad and \quad h = \frac{(m-l)}{n}.$$

Let

$$R_i(\lambda) = a_i \cos\tau(\lambda - \lambda_{i-1/2}) + b_i \sin\tau(\lambda - \lambda_{i-1/2}) + c_i, \tag{2.1}$$

be a non-polynomial spline defined on closed interval *[l,m]* reduces to polynomial spline which is quadratic as $\tau \longrightarrow 0$ and $\tau > 0$.
To calculate a_i, b_i and c_i, we define

$$\begin{aligned} R_i(\lambda_i) &= z_i,\ R_i'(\lambda_{i-1/2}) = P_{i-1/2}, \\ R_i''(\lambda_i) &= D_i,\ \ 0 \le i \le n-1. \end{aligned} \tag{2.2}$$

Using above interpolatory conditions we get

$$\begin{aligned} a_i &= -\frac{1}{\tau^2} D_i \sec\left(\frac{\vartheta}{2}\right) - \frac{1}{\tau} P_{i-1/2} \tan(\frac{\vartheta}{2}), \\ b_i &= \frac{1}{\tau} P_{i-1/2}, \\ c_i &= z_{i+1} - \frac{1}{\tau^2} D_i, \end{aligned}$$

where, $\vartheta = \tau h$.
Using continuity conditions, $R_{i-1}^{(m)}(\lambda_{i-1/2}) = R_i^{(m)}(\lambda_{i-1/2}), m = 0, 1$ we get the expression as follows:

$$z_{i-1} - 2z_i + z_{i+1} = h^2(\delta D_{i-1} + \eta D_i + \zeta D_{i+1}), 0 \le i \le n-1 \tag{2.3}$$

where,

$$\begin{aligned} \delta &= \frac{\sec(\frac{\vartheta}{2}) - 1}{\vartheta^2}, \\ \eta &= \frac{4\sec(\frac{\vartheta}{2})(1 - \cos^2(\frac{\vartheta}{2})) + 2(1 - \sec(\frac{\vartheta}{2}))}{\vartheta^2}, \\ \zeta &= \delta. \end{aligned}$$

When $\tau \to 0$, it means $\vartheta \to 0$, then $(\delta, \eta, \zeta) \to (1/8, 6/8, 1/8)$, and the scheme given by (2.3) reduces into polynomial quadratic spline relation as:

$$z_{i-1} - 2z_i + z_{i+1} = \frac{1}{8} h^2(D_{i-1} + 6D_i + D_{i+1}), 0 \le i \le n-1. \tag{2.4}$$

3 Application of the scheme

We consider a SPBVP of the form

$$z_\kappa - \epsilon_d z_{\lambda\lambda} + \epsilon_c r(\lambda) z_\lambda + s(\lambda) z = g(\lambda, \kappa), \quad (\lambda, \kappa) \in Q_T, \tag{3.1}$$

where, $Q_T = S \times I,\ S \equiv r : l < r < m,\ \partial S \equiv \{l\} \cup \{m\},\ I \equiv (0, T)$ and $l(> 0), m(> 0) \in \mathbb{R}$. The above equation with

$$z = 0,\ (\lambda, \kappa) \in \partial S \times I,$$

and

$$z(\lambda, 0) = z_0(\lambda),\ \lambda \in S.$$

3

Here, we use the following derivative approximations of higher order as:

$$
\begin{aligned}
z'_i &= \frac{z_{i+1}-z_{i-1}}{2h},\\
z'_{i-1} &= \frac{-3z_{i-1}+4z_i-z_{i+1}}{2h},\\
z'_{i+1} &= \frac{z_{i-1}-4z_i+3z_{i+1}}{2h},\\
z^j_{ti} &= \frac{z^{j+1}_i-z^j_i}{k},\\
z^j_{ti-1} &= \frac{z^{j+1}_{i-1}-z^j_{i-1}}{k},\\
z^j_{ti+1} &= \frac{z^{j+1}_{i+1}-z^j_{i+1}}{k}.
\end{aligned}
$$

We consider the following ordinary differential equation

$$
\begin{aligned}
\epsilon_d\frac{d^2z}{d\lambda^2} &= \epsilon_c r(\lambda)\frac{dz}{d\lambda}+s(\lambda)z-g(\lambda)\\
&\equiv G(\lambda,z,z'). \qquad (3.2)
\end{aligned}
$$

After implementing scheme (2.3) on BVP (3.2), we obtain:

$$
z_{i-1}-2z_i+z_{i+1} = h^2(\delta G_{i-1}+\eta G_i+\zeta G_{i+1}),\ 1\le i\le n-1 \qquad (3.3)
$$

where,

$$
\begin{aligned}
G_{i-1} &= G(\lambda_{i-1},z_{i-1},z'_{i-1}),\\
G_i &= G(\lambda_i,z_i,z'_i),\\
G_{i+1} &= G(\lambda_{i+1},z_{i+1},z'_{i+1}),
\end{aligned}
$$

Using derivative approximations, we obtain

$$
\tilde{A}z_{i-1}+\tilde{B}z_i+\tilde{C}z_{i+1} = -h^2(\delta g_{i-1}+\eta g_i+\zeta g_{i+1}), 1\le i\le n-1 \qquad (3.4)
$$

where,

$$
\begin{aligned}
\tilde{A} &= \epsilon_d+\frac{3}{2}h\delta\epsilon_c r_{i-1}-\delta h^2 s_{i-1}+\frac{h}{2}\eta\epsilon_c r_i-\frac{h}{2}\zeta\epsilon_c r_{i+1},\\
\tilde{B} &= -2\epsilon_d-2h\delta\epsilon_c r_{i-1}-\eta h^2 s_i+2h\zeta\epsilon_c r_{i+1},\\
\tilde{C} &= \epsilon_d+\frac{1}{2}h\delta\epsilon_c r_{i-1}-\zeta h^2 s_{i+1}-\frac{h}{2}\eta\epsilon_c r_i-\frac{3h}{2}\zeta\epsilon_c r_{i+1}.
\end{aligned}
$$

For solving parabolic equation (3.1) we obtain the two level spline scheme by replacing z_i by $\frac{1}{2}(z^{j+1}_i+z^j_i)$, z_{i+1} by $\frac{1}{2}(z^{j+1}_{i+1}+z^j_{i+1})$, z_{i-1} by $\frac{1}{2}(z^{j+1}_{i-1}+z^j_{i-1})$, g_i by $(\frac{z^{j+1}_i-z^j_i}{k}+g^j_i)$, g_{i+1} by $(\frac{z^{j+1}_{i+1}-z^j_{i+1}}{k}+g^j_{i+1})$ and g_{i-1} by $(\frac{z^{j+1}_{i-1}-z^j_{i-1}}{k}+g^j_{i-1})$ in (3.4) and hence we obtain as follows:

$$
\begin{aligned}
A_1z^{j+1}_{i-1}+A_2z^{j+1}_i+A_3z^{j+1}_{i+1} &= A_4z^j_{i-1}+A_5z^j_i+A_6z^j_{i+1}-h^2(\delta g^j_{i-1}+\eta g^j_i+\zeta g^j_{i+1}),\\
&\quad 1\le i\le n-1 \qquad (3.5)
\end{aligned}
$$

4

where,

$$\begin{aligned}
A_1 &= \frac{-h^2\delta}{k} + \frac{1}{2}(\epsilon_d + \frac{3}{2}h\delta\epsilon_c r_{i-1} - \delta h^2 s_{i-1} + \frac{h}{2}\eta\epsilon_c r_i - \frac{h}{2}\zeta\epsilon_c r_{i+1}),\\
A_2 &= \frac{-h^2\eta}{k} + \frac{1}{2}(-2\epsilon_d - 2h\delta\epsilon_c r_{i-1} - \eta h^2 s_i + 2h\zeta\epsilon_c r_{i+1}),\\
A_3 &= \frac{-h^2\zeta}{k} + \frac{1}{2}(\epsilon_d + \frac{1}{2}h\delta\epsilon_c r_{i-1} - \zeta h^2 s_{i+1} - \frac{h}{2}\eta\epsilon_c r_i - \frac{3h}{2}\zeta\epsilon_c r_{i+1}),\\
A_4 &= \frac{-h^2\delta}{k} + \frac{1}{2}(-\epsilon_d - \frac{3}{2}h\delta\epsilon_c r_{i-1} + \delta h^2 s_{i-1} - \frac{h}{2}\eta\epsilon_c r_i + \frac{h}{2}\zeta\epsilon_c r_{i+1}),\\
A_5 &= \frac{-h^2\eta}{k} + \frac{1}{2}(2\epsilon_d + 2h\delta\epsilon_c r_{i-1} + \eta h^2 s_i - 2h\zeta\epsilon_c r_{i+1}),\\
A_6 &= \frac{-h^2\zeta}{k} + \frac{1}{2}(-\epsilon_d - \frac{1}{2}h\delta\epsilon_c r_{i-1} + \zeta h^2 s_{i+1} + \frac{h}{2}\eta\epsilon_c r_i + \frac{3h}{2}\zeta\epsilon_c r_{i+1}).
\end{aligned}$$

Error

Here, we expand the scheme(3.5) in terms of $z(\lambda_i, \kappa_j)$ using Taylor's series and get the expression for truncation error as follows:

$$\begin{aligned}
t_i = &\Big[h^2[1-(\delta+\eta+\zeta)]D_\lambda^2 - \frac{1}{2}k[\delta h^2 s_{i-1} + \eta h^2 s_i + \zeta h^2 s_{i+1}]D_t + h^3[\delta-\zeta]D_\lambda^3\\
&+h^4[\frac{1}{12} - \frac{\delta+\zeta}{2}]D_\lambda^4 - \frac{1}{4}k^2[\delta h^2 s_{i-1} + \eta h^2 s_i + \zeta h^2 s_{i+1}]D_t^2 + h^5[\frac{\delta-\zeta}{3!}]D_\lambda^5\\
&+h^6[\frac{1}{360} - \frac{\delta+\zeta}{24}]D_\lambda^6 + ...\Big]z_i^j, \ 1 \le i \le n-1. \qquad (3.6)
\end{aligned}$$

For $\delta+\eta+\zeta = 1$ and $\delta = \zeta$, the method of $O(k+h^2)$ is obtained.

4 Stability Analysis

Here, we obtain the expression which gives information regarding stability of the scheme (3.5). We take Z_i^j as actual solution which satisfies the equation

$$\begin{aligned}
A_1 Z_{i-1}^{j+1} + A_2 Z_i^{j+1} + A_3 Z_{i+1}^{j+1} &= A_4 Z_{i-1}^j + A_5 Z_i^j + A_6 Z_{i+1}^j - h^2(\delta g_{i-1}^j + \eta g_i^j\\
&+\zeta g_{i+1}^j), \ 1 \le i \le n-1. \qquad (4.1)
\end{aligned}$$

We assume that an error $e_i^j = Z_i^j - z_i^j$ exist at each point (λ_i, κ_j), then by subtracting (3.5) from (4.1) we get the expression as

$$\begin{aligned}
A_1 e_{i-1}^{j+1} + A_2 e_i^{j+1} + A_3 e_{i+1}^{j+1} &= A_4 e_{i-1}^j + A_5 e_i^j + A_6 e_{i+1}^j,\\
&1 \le i \le n-1. \qquad (4.2)
\end{aligned}$$

To derive stability analysis for the scheme (3.5), we assume that the solution of the homogeneous part of (4.2) is of the form $e_i^j = \varpi^j e^{i\rho}$, where $\varpi \in \mathbb{C}$, $i = \sqrt{-1}$

and $\rho \in \mathbb{R}$. Finally, we get the amplification factor as

$$\varpi = \frac{A_4 e^{-i\rho} + A_5 + A_6 e^{i\rho}}{A_1 e^{-i\rho} + A_2 + A_3 e^{i\rho}}, \tag{4.3}$$

then,

$$\varpi = \frac{-\frac{h^2}{k\epsilon_d}(\delta+\rho) - \frac{h^2}{\epsilon_d}(\delta q_{i-1} - \eta q_i + \zeta q_{i+1}) + 2B_1 \sin^2(\frac{\rho}{2}) + iB_2 \sin(\frac{\rho}{2})}{-\frac{h^2}{k\epsilon_d}(\delta+\rho) + \frac{h^2}{\epsilon_d}(\delta q_{i-1} - \eta q_i + \zeta q_{i+1}) - 2B_1 \sin^2(\frac{\rho}{2}) + iB_2 \sin(\frac{\rho}{2})},$$

where

$$B_1 = 1 + \frac{h^2}{k\epsilon_d}(\delta+\zeta) + h\frac{\epsilon_c}{\epsilon_d}(\delta p_{i-1} - 2\zeta p_{i+1}) + h^2 \frac{1}{\epsilon_d}(\delta q_{i-1} + \zeta q_{i+1}),$$

$$B_2 = \frac{h^2}{k\epsilon_d}(\delta-\rho) + \frac{1}{2\epsilon_d}[h\epsilon_c(\delta p_{i-1} + \eta p_i - \zeta p_{i+1}) + h^2(\zeta q_{i+1} - \delta q_{i-1})].$$

The condition for the scheme to be stable is $|\varpi| \leq 1$. As we know that $0 \leq \sin^2(\frac{\rho}{2}) \leq 1$ and $\epsilon_d \propto h$, then from above relation it is easily verified that $|\varpi| \leq 1$ for every ρ. Hence the developed method is unconditionally stable.

5 Numerical Illustrations

We consider three second order SPBVPs. The maximum absolute errors(MAE) are tabulated in Tables 1-4 depending upon the choice of parameters. The convergence rate is denoted by α_n and is computed by following expression:

$$\alpha_n = \ln_2(Er_{n,k}/Er_{2n,k}),$$

and there is a different way to find rate of convergence denoted by $\tilde{\alpha}_n$ and is computed by using

$$\tilde{\alpha}_n = \ln_2(Er_{n,k}/Er_{2n,k/2}).$$

Example 1:

Consider the following problem from Zahra et al.[10].

$$z_\kappa - \epsilon_d\, z_{\lambda\lambda} + z_\lambda = g(\lambda,\kappa), T = 1,$$

in [0,1] associated with $z(0,\kappa) = 0, z(1,\kappa) = 0$ and $z_0(\lambda) = \exp(-1/\epsilon_d) + (1 - \exp(-1/\epsilon_d))\lambda - \exp(-(1-\lambda)/\epsilon_d)$, where
$g(\lambda,\kappa) = \exp(-\kappa)(-c_1 + c_2(1-\lambda) + \exp(-(1-\lambda)/\epsilon_d))$.

The analytical solution is $z(\lambda,\kappa) = \exp(-\kappa)(c_1 + c_2\lambda - \exp(-(1-\lambda)/\epsilon_d))$, where $c_1 = \exp(-1/\epsilon_d), c_2 = 1 - \exp(-1/\epsilon_d)$. The numerical results for $N = 2^4, 2^5, 2^6, 2^7$ and $\epsilon_d = 1/2^8, 1/2^{10}, 1/2^{12}, 1/2^{24}, 1/2^{26}$ using parameters $(\delta,\eta,\zeta) = \frac{1}{8}(1,6,1)$ compared with Zahra et al.[10] are tabulated in Table 1. And for $N = 2^4, 2^5, 2^6, 2^7, 2^8, 2^9$ and $\epsilon_d = 1, 1/4, 1/16, 1/64$ using parameters $(\delta,\eta,\zeta) = \frac{1}{8}(1,6,1)$ compared with Clavero et al.[2] are tabulated in Table 2.

Example 2:

Consider the following PDE from Zahra et al.[10]

$$z_\kappa - \epsilon_d\ z_{\lambda\lambda} + \epsilon_c\ z_\lambda \quad = \quad g(\lambda,\kappa),\ T = 1,$$

in [0,1] associated with $z(0,\kappa) = 0, z(1,\kappa) = 0$ and $z_0(\lambda) = [\phi_1\cos(\pi\lambda) + \phi_2\sin(\pi\lambda) + \psi_1\exp(\theta_1\lambda) + \psi_2\exp(-\theta_2(1-\lambda))]$, where
$g(\lambda,\kappa) = \exp(-\kappa)[\{-\phi_1\cos(\pi\lambda) - \phi_2\sin(\pi\lambda) - \psi_1\exp(\theta_1\lambda) - \psi_2\exp(-\theta_2(1-\lambda))\} + \epsilon_d\{\phi_1\pi^2\cos(\pi\lambda) + \phi_2\pi^2\sin(\pi\lambda) - \frac{\psi_1}{\theta_1^2}\exp(\theta_1\lambda) - \frac{\psi_2}{\theta_2^2}\exp(-\theta_2(1-\lambda))\} + \epsilon_c\{-\phi_1\pi\cos(\pi\lambda) + \phi_2\pi\sin(\pi\lambda) + \frac{\psi_1}{\theta_1}\exp(\theta_1\lambda) + \frac{\psi_2}{\theta_2}\exp(-\theta_2(1-\lambda))\}]$. The analytical solution is $z(\lambda,\kappa) = \exp(-\kappa)[\phi_1\cos(\pi\lambda) + \phi_2\sin(\pi\lambda) + \psi_1\exp(\theta_1\lambda) + \psi_2\exp(-\theta_2(1-\lambda))]$ where,

$$\begin{aligned}
\phi_1 &= \frac{\epsilon_d\pi^2 + 1}{\epsilon_c^2\pi^2 + (\epsilon_d\pi^2+1)^2}\ , \\
\phi_2 &= \frac{\epsilon_c\pi}{\epsilon_c^2\pi^2 + (\epsilon_d\pi^2+1)^2}\ , \\
\psi_1 &= -\phi_1\frac{1+\exp(-\theta_2)}{1-\exp(\theta_1-\theta_2)}\ , \\
\psi_2 &= \phi_1\frac{1+\exp(\theta_1)}{1-\exp(\theta_1-\theta_2)}\ , \\
\theta_1 &= \frac{\epsilon_c - \sqrt{\epsilon_c^2 + 4\epsilon_d}}{2\epsilon_d}\ , \\
\theta_2 &= \frac{\epsilon_c + \sqrt{\epsilon_c^2 + 4\epsilon_d}}{2\epsilon_d}\ .
\end{aligned}$$

The numerical results for $N = 2^4, 2^5, 2^6, 2^7, 2^8,\ \epsilon_d = 1, 1/4, 1/16$ and $\epsilon_c = 10^{-3}, 10^{-4}, 10^{-5}$ using parameters $(\delta,\eta,\zeta) = \frac{1}{8}(1,6,1)$ are tabulated in Table 3.

Example 3:

Consider the following PDE from Mohanty et al.[6]

$$\epsilon_d\ z_{\lambda\lambda} - z_\kappa + \frac{1}{\lambda}z_\lambda \quad = \quad g(\lambda,\kappa),\ 0 \le \lambda \le 1,\ \kappa > 0$$

The analytical solution is $z(\lambda,\kappa) = \exp(-\kappa)\sinh\lambda$. The right-hand-side functions, initial and boundary conditions may be obtained using the actual solution given above as a test procedure. The numerical results for $N = 2^4, 2^5, 2^6, 2^7, 2^8$ and $\epsilon_d = 1/2, 1/8, 1/16, 1/32, 1/64, 1/128$ using parameters $(\delta,\eta,\zeta) = \frac{1}{12}(1,10,1)$ compared with Mohanty et al.[6] are tabulated in Table 4.

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Table 1: MAE of example 1 for $(\delta, \eta, \zeta) = \frac{1}{8}(1, 6, 1)$

Method	$N \backslash \epsilon_d$	$1/2^8$	$1/2^{10}$	$1/2^{12}$	$1/2^{24}$	$1/2^{26}$
Presented method	2^4	1.2136×10^{-02}	1.3494×10^{-02}	1.3835×10^{-02}	1.3949×10^{-02}	1.3949×10^{-02}
α_n		1.1344	0.9591	0.9231	0.91170	0.91170
Zahra et al.[10]		3.5638×10^{-02}	5.1972×10^{-02}	6.7088×10^{-02}	7.3818×10^{-02}	7.1839×10^{-02}
Presented method	2^5	5.5284×10^{-03}	6.9407×10^{-03}	7.2961×10^{-03}	7.4147×10^{-03}	7.4147×10^{-03}
α_n		1.4946	1.0549	0.9785	0.9554	0.9554
Zahra et al.[10]		1.300×10^{-02}	1.4234×10^{-02}	2.3185×10^{-02}	3.4126×10^{-02}	3.4128×10^{-02}
Presented method	2^6	1.9619×10^{-03}	3.3406×10^{-03}	3.7029×10^{-03}	3.8238×10^{-03}	3.8238×10^{-03}
α_n		2.1590	1.1997	1.0249	0.9776	0.9776
Zahra et al.[10]		9.3378×10^{-03}	8.3305×10^{-03}	8.2129×10^{-03}	1.5756×10^{-02}	1.5761×10^{-02}
Presented method	2^7	4.4173×10^{-04}	1.4543×10^{-03}	1.8198×10^{-03}	1.9419×10^{-03}	1.9419×10^{-03}
Zahra et al.[10]		8.4218×10^{-03}	7.9579×10^{-03}	7.6243×10^{-03}	9.1052×10^{-03}	9.1078×10^{-03}

Table 2: MAE of example 1 for $(\delta, \eta, \zeta) = \frac{1}{8}(1, 6, 1)$

Method	$N \setminus \epsilon_d$	1	1/4	1/16	1/64
Presented Method	2^4	7.8074×10^{-04}	1.2280×10^{-03}	1.0487×10^{-02}	1.0921×10^{-01}
Clavero et al.[2]		1.3076×10^{-03}	1.7398×10^{-03}	4.0133×10^{-02}	5.9664×10^{-02}
	$\tilde{\alpha}_n$	1.7534	1.9008	2.5426	1.3506
Presented Method	2^5	2.3156×10^{-04}	3.2887×10^{-04}	1.8000×10^{-03}	4.2823×10^{-02}
Clavero et al.[2]		7.9078×10^{-04}	9.6845×10^{-03}	2.5552×10^{-03}	3.7372×10^{-02}
	$\tilde{\alpha}_n$	1.8952	1.8851	2.4931	2.0663
Presented Method	2^6	6.2255×10^{-05}	8.9033×10^{-05}	3.1971×10^{-04}	1.0225×10^{-02}
Clavero et al.[2]		3.6986×10^{-04}	5.1056×10^{-03}	1.5865×10^{-02}	2.1792×10^{-02}
	$\tilde{\alpha}_n$	1.9602	1.9124	2.4935	2.6020
Presented Method	2^7	1.5998×10^{-05}	2.3652×10^{-05}	5.6774×10^{-05}	1.6841×10^{-03}
Clavero et al.[2]		1.8894×10^{-04}	2.6223×10^{-03}	9.5603×10^{-03}	1.2381×10^{-03}
	$\tilde{\alpha}_n$	1.9638	1.9347	2.4473	2.5951
Presented Method	2^8	4.1011×10^{-06}	6.1867×10^{-06}	1.0409×10^{-05}	2.7872×10^{-04}
Clavero et al.[2]		9.5517×10^{-05}	1.3289×10^{-03}	5.5999×10^{-03}	6.9704×10^{-03}
	$\tilde{\alpha}_n$	1.9676	1.9514	2.3371	2.6674
Presented Method	2^9	1.0486×10^{-06}	1.5996×10^{-06}	2.0601×10^{-06}	4.3874×10^{-05}
Clavero et al.[2]		4.8028×10^{-05}	6.6891×10^{-04}	3.2019×10^{-03}	3.9052×10^{-03}

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Table 3: MAE of example 2 for $(\delta, \eta, \zeta) = \frac{1}{8}(1, 6, 1)$

ϵ_d	1			1/4			1/16		
$\epsilon_c \setminus N$	10^{-3}	10^{-4}	10^{-5}	10^{-3}	10^{-4}	10^{-5}	10^{-3}	10^{-4}	10^{-5}
2^4	5.7028×10^{-06}	5.6372×10^{-06}	5.6328×10^{-06}	7.6099×10^{-03}	7.5937×10^{-03}	7.5920×10^{-03}	3.0375×10^{-01}	3.0260×10^{-01}	3.0248×10^{-01}
α_n	1.4451	1.4479	1.4482	1.8084	1.8084	1.8084	1.8258	1.8258	1.8258
2^5	2.0944×10^{-06}	2.0671×10^{-06}	2.0643×10^{-06}	2.1726×10^{-03}	2.1681×10^{-03}	2.1676×10^{-03}	8.5684×10^{-02}	8.5356×10^{-02}	8.5324×10^{-02}
α_n	1.8444	1.8459	1.8461	1.8568	1.8568	1.8568	1.9140	1.9140	1.9140
2^6	5.8324×10^{-07}	5.7502×10^{-07}	5.7420×10^{-07}	5.9986×10^{-04}	5.9857×10^{-04}	5.9845×10^{-04}	2.2736×10^{-02}	2.2650×10^{-02}	2.2640×10^{-02}
α_n	1.9487	1.9496	1.9497	1.9306	1.9306	1.9306	1.9572	1.9572	1.9572
2^7	1.5108×10^{-07}	1.4886×10^{-07}	1.4864×10^{-07}	1.5736×10^{-04}	1.5702×10^{-04}	1.5698×10^{-04}	5.8551×10^{-03}	5.8328×10^{-03}	5.8800×10^{-03}
α_n	1.9810	1.9815	1.9816	1.9658	1.9658	1.9658	1.9786	1.9786	1.9786
2^8	3.8272×10^{-08}	3.7696×10^{-08}	3.7639×10^{-08}	4.0284×10^{-05}	4.0197×10^{-05}	4.0188×10^{-05}	1.4856×10^{-03}	1.4799×10^{-03}	1.4799×10^{-03}

Table 4: MAE of example 3 for $(\delta, \eta, \zeta) = \frac{1}{12}(1, 10, 1)$

Method	$N \setminus \epsilon_d$	1/2	1/8	1/16	1/32	1/64	1/128
Presented method	2^4	7.2294×10^{-04}	8.0022×10^{-04}	8.2241×10^{-04}	8.3576×10^{-04}	8.4320×10^{-04}	8.4714×10^{-04}
Mohanty et al.[6]		0.2924×10^{-03}	0.4454×10^{-03}	0.4777×10^{-03}	0.5054×10^{-03}	0.5344×10^{-03}	0.5615×10^{-03}
Presented method	2^5	1.2613×10^{-05}	1.3899×10^{-05}	1.4267×10^{-04}	1.4488×10^{-04}	1.4610×10^{-04}	1.4675×10^{-04}
Mohanty et al.[6]		0.7286×10^{-04}	0.1129×10^{-03}	0.1239×10^{-03}	0.1410×10^{-03}	0.1869×10^{-03}	0.3134×10^{-03}
Presented method	2^6	2.2181×10^{-05}	2.4391×10^{-05}	2.5022×10^{-05}	2.5721×10^{-05}	2.5610×10^{-05}	2.5721×10^{-05}
Mohanty et al.[6]		0.1814×10^{-04}	0.2835×10^{-04}	0.3166×10^{-04}	0.3984×10^{-04}	0.9429×10^{-04}	0.1684×10^{-03}
Presented method	2^7	3.9130×10^{-06}	4.2982×10^{-06}	4.4079×10^{-06}	4.5295×10^{-06}	4.5102×10^{-06}	4.5295×10^{-06}
Mohanty et al.[6]		0.4524×10^{-05}	0.7091×10^{-05}	0.7987×10^{-05}	0.1088×10^{-04}	0.4743×10^{-04}	0.9120×10^{-04}
Presented method	2^8	6.9111×10^{-07}	7.5878×10^{-07}	7.7796×10^{-07}	7.9929×10^{-07}	7.7590×10^{-07}	7.9929×10^{-07}
Mohanty et al.[6]		0.1129×10^{-05}	0.1771×10^{-05}	0.2002×10^{-05}	0.2844×10^{-05}	0.1821×10^{-05}	0.5283×10^{-04}

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Conclusion

We have presented two level scheme using non-polynomial spline for solving singularly perturbed parabolic equations based on one dimension. In examples 1, 2 and 3, we have computed maximum absolute errors for different values of N and ϵ_d for the sake of comparison with references [2,6,10] and results are tabulated in Tables 1-4. From tables it is shown that our method is much better in accuracy than the methods given by Clavero et al.[2], Mohanty et al.[6] and Zahra et al.[10]. It has already been proved that the presented algorithm gives higher numerical rate of convergence. It has also shown that the scheme is unconditionally stable.

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Solving System of Boundary Value Problems using Non polynomial Spline Methods Based on Off-step Mesh

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Abstract

We present two non polynomial spline methods based on quasi-variable mesh using off-step points to solve the system of boundary value problems which are nonlinear. We also discuss how the methods handle the presence of singularity. The proposed methods has been shown second and third-order convergent for a model linear problem. The methods are implemented on existing problems which are linear, non linear as well as singular. The obtained numerical results approximate the exact solutions very well and validate the theoretical findings.

Key words: Off-step, non polynomial, quasi-variable mesh, singular, nonlinear, system.
Mathematics Subject Classification(2010): 65L10.

1 Introduction

In this paper, we seek solution for the following system of M boundary value problems(BVPs) which are non linear as well as singular.

$$\frac{d^2y^i}{dx^2} = f^i(x, y^1, ..., y^i, ..., y^M, \frac{dy^1}{dx}, ..., \frac{dy^i}{dx}, ..., \frac{dy^M}{dx}), \tag{1.1}$$
$$y^i(0) = a_i, y^i(1) = b_i, \quad \text{where} \quad a_i, b_i \in R, \ i = 1(1)M. \tag{1.2}$$

We consider $-\infty < y^i, \frac{dy^i}{dx} < \infty$ and the conditions such that f^i is continuous and its partial derivatives w.r.t. y^j and $\frac{dy^i}{dx}$ exist, continuous and are positive. Also partial derivative w.r.t. $\frac{dy^i}{dx}$ is bounded by some $K > 0, \ j, i = 1(1)M$, to ensure the existence [13] of a unique solution $(1.1) - (1.2)$.

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Such systems like the fourth order Euler differential equations [5], coupled Navier stokes in fluid dynamics and Maxwell's equations of electromagnetism[9], system of differential equations[28], fourth order non linear differential equations [27] simulates many real world problems. A few more examples are as follows:
(i)In Plate deflection theory

$$(-1)^n y^{(2n)}(x) = f(x, y(x)), n \in N$$
$$y^{(2i)}(a) = A_{2i}, y^{(2i)}(b) = B_{2i}, i \in [0, k-1]$$

(ii)Three box cars on a level track connected by springs is modelled as follows:

$$mx_1'' = -sx_1 + sx_2,$$
$$mx_2'' = sx_1 - 2sx_2 + sx_3,$$
$$mx_3'' = -sx_3,$$

where m, x_1, x_2, x_3 and s are masses,positions of the boxcars and Hooke's constant.

(iii)A horizontal earthquake wave F affects every floor of a building. If there are three floors, then equations for the floor is modelled as follows:

$$M_1x_1'' = -(r_1 + r_2)x_1 + r_2x_2,$$
$$M_2x_2'' = r_2x_1 - (r_2 + r_3)x_2 + r_3x_3,$$
$$M_3x_3'' = r_3x_2 - (r_3 + r_4)x_3,$$

where M_i, x_1, x_2, x_3 and r_i are point masses of each floor, location of masses and Hooke's constant.

Such systems of BVPs comprising first or second order BVPs not only models many real life problems but are also instrumental in solving many higher order problems by decomposing them. Authors like Aftabizadeh[1], Agarwal[2], Regan[24] have developed theories related to existence and uniqueness of solutions for these BVPs. But, for our work, we focus on system of second order BVPs which are non linear as well singular in nature. These problems have extensive application and has been the cause of interest for many authors. Many efficient numerical methods have been developed to solve second order BVPs and 'Splines' have been very instrumental for solving such problems. Mohanty et.al.([15],[16], [18], [21]) developed AGE iterative methods. In these methods, using Taylor's theorem derivatives are approximated and accordingly a finite difference scheme was developed. Then the resultant system solved by splitting the coefficient matrix into sum of three matrices. Also Mohanty et.al.([17],[19],[20],[22]) derived polynomial and non polynomial spline methods based on uniform and variable mesh to solve class of problems ranging from linear, nonlinear, singular and singularly perturbed BVP. A third order cubic spline method based on non uniform mesh was developed by Kadalbajoo et. al[12] to solve singularly perturbed BVPs. BVPs of eighth order were solved by Akram and Rehman[4] using kernel space method. Eighth and sixth order BVPs were solved by Siddiqi and Akram ([30], [31]) using non-polynomial and

septic spline. Jha and Bieniasz[11] developed a scheme based on geometric mesh to solve sixth order differential equation by converting it into system of second order differential equations. Infact, very recently, apart from the schemes based on classical finite differences some other kinds of methods were also developed. Bhrawy et. al.[6] developed collocation method based on Jacobi polynomials and solved nonlinear second-order initial value problems. Dwivedi and Singh [7] developed collocation method based on Fibbonacci polynomial to solve sub diffusion equations. Singh et.al.[32] developed finite difference scheme based on homotopy analysis transform technique to solve fractional non-linear coupled problem. Such considerable amount of work has motivated us to develop a numerical method to solve the higher order problem as well as system of linear and non linear singular BVPs.

In this paper, generalized non polynomial spline schemes have been developed which are based on off-step points using quasi-variable mesh. We use a second order BVP to derive the methods. As per the methods developed, we decompose the higher order BVP into system of second order BVPs (1) alongwith modifying the boundary conditions. Also, we have solved singular BVPs. The off-step points used in the method allows us to overcome the singularity. Moreover, since we use the quasi-variable mesh the error gets uniformly distributed throughout the solution domain. Finally, as we use the boundary conditions in the scheme, we get a tri-diagonal matrix with block elements representing the system of equations to be solved.

We have solved seven problems and demonstrated the accuracy of the proposed methods. The BVPs considered in this paper have been solved by other methods as well. Twizell [29] used modified extrapolation method to solve fourth order linear BVPs, Akram and Siddiqi[3] used non polynomial spline method which is second order convergent to solve linear sixth order BVPs. Khan and Khandelwal[14]and Sakai and Usmani[25] used splines to solve nonlinear fourth and sixth order BVPs.

2 Method Formulation

We use a non linear BVP of second order and derive the method in scalar form:

$$y'' \quad = \quad f(x, y, y'), \text{ subject to } y(0) = a, y(1) = b. \tag{2.1}$$

Now, we divide the solution region [0,1] into $N+1$ points such as $x_j, j = 0(1)N$ with mesh size h_j such that $x_j = x_{j-1} + h_j, \frac{h_{j+1}}{h_j} = \sigma_j, j = 1(1)N-1$ where the σ_j is the mesh ratio. When mesh ratio is one, the quasi-variable mesh converts to a uniform mesh with width, say h. Now, we choose $\sigma_j = \sigma$ a constant $\forall j$ without loss of generality. Also, let the exact solution of (2.1) be $y(x_j)$ or y_j at the grid points x_j. Now, we define the the following non polynomial spline function:

$$S_j(x) = d_j \sin(kx - kx_j) + c_j \cos(kx - kx_j) + b_j(x - x_j) + a_j, \; x_{j-1} \le x \le x_j. \tag{2.2}$$

Here, $S_j(x)$ has continuous second derivative in $[0,1]$ and $S_j(x), S_j'(x)$ interpolates at the mesh points x_j. Using the definition of the spline, we determine values for the unknowns a_j, b_j, c_j and d_j as:

$$a_j = y_j + \frac{f_j}{k^2}, \tag{2.3}$$

$$b_j = \frac{f_j - f_{j-\frac{1}{2}}}{k^2 h_j} - \frac{y_{j-1} - y_j}{h_j},$$

$$c_j = -\frac{f_j}{k^2},$$

$$\text{and} \quad d_j = -\frac{f_k \cos\theta_j - f_{j-\frac{1}{2}}}{k^2 \sin kh_j}. \tag{2.4}$$

Using the spline's first derivative continuity conditions, we get the non polynomial spline method based on off-step points as:

$$\sigma y_{j-1} - y_j(1+\sigma) + y_{j+1} = h_j^2(P\sigma f_{j-\frac{1}{2}} + Q\sigma f_j + R\sigma f_{j+\frac{1}{2}}) + T_j^3, \tag{2.5}$$

where

$$R = \frac{2kh_{j+1} - \sin kh_{j+1}}{2k^2 h_{j+1} \sin kh_{j+1}}, P = \frac{kh_j - \sin kh_j \cos kh_j}{kh_j \sin kh_j}, \tag{2.6}$$

$$Q = \frac{2(\sigma_j+1)(\cos(kh_j\sigma - kh_j) - \cos(kh_j\sigma + kh_j)) - 2\sigma kh_j \sin(kh_j\sigma + kh_j)}{(kh_j)^2(\cos(kh_j\sigma - kh_j) - \cos(kh_j\sigma + kh_j))} \tag{2.7}$$

Now, we also derive the consistency condition using (2.5) – (2.7) i.e.,

$$\tan(\frac{kh_j}{2}) + \tan(\frac{kh_{j+1}}{2}) = \frac{kh_j}{2} + \frac{kh_{j+1}}{2}. \tag{2.8}$$

We solve equation (2.8) for kh_j and consider the non-zero smallest positive root $kh_j = 8.98681891$. But, with this, the order of error term T_j^3 in (2.5) remains four. Now, we derive another off-step method using Taylor's expansion for $j = 1(1)N-1$ as

$$y_{j-1} - y_j(1+\sigma) + \sigma y_{j+1} = h_j^2\sigma(Af_{j-\frac{1}{2}} + Bf_{j+\frac{1}{2}}) + T_j^2, \tag{2.9}$$

$$\text{for } A = \frac{(2+\sigma)}{6}, \; B = \frac{(2\sigma+1)}{6}. \tag{2.10}$$

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Now, the following approximations are defined at $x_j, j = 1(1)N-1$,

$$S_j = (\sigma+1)\sigma, \tag{2.11}$$

$$\bar{y}_{j+\frac{1}{2}} = \frac{y_j + y_{j+1}}{2}, \tag{2.12}$$

$$\bar{y}_{j-\frac{1}{2}} = \frac{y_j + y_{j-1}}{2}, \tag{2.13}$$

$$\bar{y'}_{j+\frac{1}{2}} = \frac{y_{j+1} - y_j}{h_j\sigma}, \tag{2.14}$$

$$\bar{y'}_{j-\frac{1}{2}} = \frac{y_j - y_{j-1}}{h_j}, \tag{2.15}$$

$$\bar{y'}_j = \frac{y_{j+1} - y_j(1-\sigma^2) - \sigma^2 y_{j-1}}{S_j h_j}, \tag{2.16}$$

$$\bar{f}_j = f(x_j, y_j, \bar{y'}_j), \tag{2.17}$$

$$\bar{f}_{j-\frac{1}{2}} = f(x_{j-\frac{1}{2}}, \bar{y}_{j-\frac{1}{2}}, \bar{y'}_{j-\frac{1}{2}}), \tag{2.18}$$

$$\bar{f}_{j+\frac{1}{2}} = f(x_{j+\frac{1}{2}}, \bar{y}_{j+\frac{1}{2}}, \bar{y'}_{j+\frac{1}{2}}). \tag{2.19}$$

Next, we define higher order approximation of y_j and y'_j to raise order of the error term T_j^3 in equation (2.5) :

$$\hat{y}_j = y_j + h_j^2\delta(\bar{f}_{j-\frac{1}{2}} + \bar{f}_{j+\frac{1}{2}}), \tag{2.20}$$

$$\hat{y'}_j = \bar{y'}_j - h_j\gamma(\bar{f}_{j-\frac{1}{2}} - \bar{f}_{j+\frac{1}{2}}), \tag{2.21}$$

where γ, δ are unknowns. This gives us the modified $\bar{f}_j$ i.e.,

$$\hat{f}_j = f(x_j, \hat{y}_j, \hat{y'}_j). \tag{2.22}$$

Now, expanding the approximations (2.12) – (2.22) we get the following:

$$P = \frac{\sigma}{3} + O(kh_j{}^2), \tag{2.23}$$

$$R = \frac{1}{3} + O(kh_j{}^2), \tag{2.24}$$

$$Q = \frac{(\sigma+1)}{6} + O(kh_j{}^2), \tag{2.25}$$

$$\bar{y}_{j+\frac{1}{2}} = y_{j+\frac{1}{2}} + \frac{(h_j\sigma)^2}{8}y''_j + O(h_j^3), \tag{2.26}$$

$$\bar{y}_{j-\frac{1}{2}} = y_{j-\frac{1}{2}} + \frac{(h_j)^2}{8}y''_j + O(h_j^3), \tag{2.27}$$

$$\bar{y'}_{j+\frac{1}{2}} = y'_{j+\frac{1}{2}} + \frac{(h_j\sigma)^2}{24}y'''_j + O(h_j^3), \tag{2.28}$$

$$\bar{y'}_{j-\frac{1}{2}} = y'_{j-\frac{1}{2}} + \frac{(h_j)^2}{24}y'''_j + O(h_j^3), \tag{2.29}$$

$$\hat{y}_j = y_j + \delta h_j^2(2y''_j) + O(h_j^3), \sigma \neq 1, \tag{2.30}$$

$$\hat{y'}_j = y'_j + \frac{h_j^2 y'''_j}{6}((1+3\gamma)\sigma + 3\gamma) + O(h_j^3), \tag{2.31}$$

$$\bar{f}_{j+\frac{1}{2}} = f_{j+\frac{1}{2}} + \frac{(h_j\sigma)^2 y''_j}{8}\frac{\partial f}{\partial y_j} + \frac{(h_j\sigma)^2 y'''_j}{24}\frac{\partial f}{\partial y'_j} + O(h_j^3), \tag{2.32}$$

$$\bar{f}_{j-\frac{1}{2}} = f_{j-\frac{1}{2}} + \frac{h_j^2 y''_j}{8}\frac{\partial f}{\partial y_j} + \frac{h_j^2 y'''_j}{24}\frac{\partial f}{\partial y'_j} + O(h_j^3), \tag{2.33}$$

$$\hat{f}_j = f_j + 2h_j^2\delta y''_j\frac{\partial f}{\partial y_j} + \frac{h_j^2}{6}(\sigma + 3\gamma(1+\sigma))y'''_j\frac{\partial f}{\partial y'_j} + O(h_j^3). \tag{2.34}$$

Thus, we develop the first method by discretizing the proposed BVP (2.1) based on the method (2.9) as:

$$\sigma y_{j-1} - (1+\sigma)y_j + y_{j+1} = h_j^2(A\sigma\bar{f}_{j-\frac{1}{2}} + B\sigma\bar{f}_{j+\frac{1}{2}}) + T_j^2. \tag{2.35}$$

In this method, we can show that for $\sigma \neq 1$, the order of truncation error T_j^2 is $O(h_j^4)$ using the approximations $(2.32)-(2.33)$. Also, if we use the off-step non-polynomial scheme(2.5) along with the approximation $(2.26)-(2.34)$, we get the second method as:

$$\begin{aligned} y_{j+1} - (1+\sigma)y_j + \sigma y_{j-1} &= h_j^2\sigma(P\bar{f}_{j-\frac{1}{2}} + Q\hat{f}_j + R\bar{f}_{j+\frac{1}{2}}) \\ &- h_j^4[(\frac{R\sigma^2 + Q4((1+3\gamma)\sigma + 3\gamma) + P}{24})y'''_j\frac{\partial f}{\partial y'_j} \\ &+ (\frac{R\sigma^2}{8} + 2Q\delta + \frac{P}{8})y''_j\frac{\partial f}{\partial y_j}] + T_j^3. \end{aligned} \tag{2.36}$$

The coefficients of order four of h_j is equated to zero to get the value of δ, γ so as to raise the order of local truncation error T_j^3. Thus, we get

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$\gamma = -\frac{(R\sigma^2+P+4Q\sigma)}{12Q(1+\sigma)}, \delta = -\frac{(R+P\sigma^2)}{16Q}$. In case of uniform mesh, the local truncation error becomes of order six. We also ensure the necessary condition for convergence of the methods provided by Jain [10], that the coefficients A, B in method (2.35)and in method (2.36) P, Q and R are positive for $\sigma > 0$. Hence, both the proposed off-step three point discretization using the approximate solutions Y_j at x_j are as follows:

$$Y_{j+1} - (\sigma+1)Y_j + \sigma Y_{j-1} = h_j^2\sigma(A\bar{F}_{j-\frac{1}{2}} + B\bar{F}_{j+\frac{1}{2}}), \tag{2.37}$$

and

$$Y_{j+1} - (\sigma+1)Y_j + \sigma Y_{j-1} = h_j^2\sigma(R\bar{F}_{j+\frac{1}{2}} + Q\hat{F}_j + P\bar{F}_{j-\frac{1}{2}}). \tag{2.38}$$

3 Generalised Methods

We develop the generalized methods by using the following approximations and scalar methods developed in the last section, thus, solving $(1.1) - (1.2)$ we get,

$$S_j = (\sigma+1)\sigma, \tag{3.1}$$

$$\bar{Y}^i_{j+\frac{1}{2}} = \frac{Y^i_j + Y^i_{j+1}}{2}, \tag{3.2}$$

$$\bar{Y}^i_{j-\frac{1}{2}} = \frac{Y^i_j + Y^i_{j-1}}{2}, \tag{3.3}$$

$$\bar{Y'}^i_{j+\frac{1}{2}} = \frac{Y^i_j - Y^i_{j+1}}{h_j\sigma}, \tag{3.4}$$

$$\bar{Y'}^i_{j-\frac{1}{2}} = \frac{Y^i_j - Y^i_{j-1}}{h_j}, \tag{3.5}$$

$$\bar{Y'}^i_j = \frac{Y^i_{j+1} - (1-\sigma^2)Y^i_j - \sigma^2 Y^i_{j-1}}{S_j h_j}, \tag{3.6}$$

$$\bar{f}^i_j = f^i(x_j, Y_j, Y^{(1)}_j, Y^{(2)}_j, ..., Y^i_j, ..., Y^{(M)}_j, \bar{Y'}^{(1)}_j, \bar{Y'}^{(2)}_j, ..., \bar{Y'}^i_j, ..., \bar{Y'}^{(M)}_j), \tag{3.7}$$

$$\bar{f}^i_{j-\frac{1}{2}} = f^i(x_{j-\frac{1}{2}}, \bar{Y}_{j-\frac{1}{2}}, \bar{Y}^{(1)}_{j-\frac{1}{2}}, \bar{Y}^{(2)}_{j-\frac{1}{2}}, ..., \bar{Y}^i_{j-\frac{1}{2}}, ..., \bar{Y}^{(M)}_{j-\frac{1}{2}}, \bar{Y'}^{(1)}_{j-\frac{1}{2}}, \bar{Y'}^{(2)}_{j-\frac{1}{2}}, ..., \bar{Y'}^i_{j-\frac{1}{2}}, ..., \bar{Y'}^{(M)}_{j-\frac{1}{2}}), \tag{3.8}$$

$$\bar{f}^i_{j+\frac{1}{2}} = f^i(x_{j+\frac{1}{2}}, \bar{Y}_{j+\frac{1}{2}}, \bar{Y}^{(1)}_{j+\frac{1}{2}}, \bar{Y}^{(2)}_{j+\frac{1}{2}}, ..., \bar{Y}^i_{j+\frac{1}{2}}, ..., \bar{Y}^{(M)}_{j+\frac{1}{2}}, \bar{Y'}^{(1)}_{j+\frac{1}{2}}, \bar{Y'}^{(2)}_{j+\frac{1}{2}}, ..., \bar{Y'}^i_{j+\frac{1}{2}}, ..., \bar{Y'}^{(M)}_{j+\frac{1}{2}}), \tag{3.9}$$

$$\hat{Y}^i_j = Y^i_j + h^2_j \delta_i (\bar{f}^i_{j+\frac{1}{2}} + \bar{f}^i_{j-\frac{1}{2}}), \tag{3.10}$$

$$\hat{Y'}^i_j = \bar{Y'}^i_j + h_j \gamma_i (\bar{f}^i_{j+\frac{1}{2}} - \bar{f}^i_{j-\frac{1}{2}}), \tag{3.11}$$

$$\hat{f}^i_j = f^i(x_j, \hat{Y}_j, \hat{Y}^{(1)}_j, \hat{Y}^{(2)}_j, ..., \hat{Y}^i_j, ..., \hat{Y}^{(M)}_j, \hat{Y'}^{(1)}_j, \hat{Y'}^{(2)}_j, ..., \hat{Y'}^i_j, ..., \hat{Y'}^{(M)}_j), \tag{3.12}$$

$$Y^i_{j+1} - (1+\sigma)Y^i_j + \sigma Y^i_{j-1} = h^2_j \sigma (A\bar{f}^i_{j-\frac{1}{2}} + B\bar{f}^i_{j+\frac{1}{2}}), \tag{3.13}$$

$$Y^i_{j+1} - (1+\sigma)Y^i_j + \sigma Y^i_{j-1} = h^2_j \sigma (R\bar{f}^i_{j+\frac{1}{2}} + Q\hat{f}^i_j + P\bar{f}^i_{j-\frac{1}{2}}), \tag{3.14}$$

where

$$A = \frac{(2+\sigma)}{6}, \; B = \frac{(2\sigma+1)}{6}, \tag{3.15}$$

$$P = \frac{kh_j - \sin kh_j \cos kh_j}{k^2 kh_j \sin kh_j}, \; R = \frac{2kh_{j+1} - \sin kh_{j+1}}{2k^2 kh_{j+1} \sin kh_{j+1}}, \tag{3.16}$$

$$Q = \frac{2(\sigma+1)(\cos(kh_j\sigma - kh_j) - \cos(kh_j\sigma + kh_j)) - 2\sigma kh_j \sin(kh_j\sigma + kh_j)}{(kh_j)^2(\cos(kh_j\sigma - kh_j) - \cos(kh_j\sigma + kh_j))} \tag{3.17}$$

4 Illustration of the Method

Consider a linear singular BVP of fourth order as follows:

$$\frac{d^4y(x)}{dx^4} = a(x)y(x) + d(x), x \neq 0, \tag{4.1}$$

$$y(0) = c_1, y(1) = d_1, \frac{d^2y}{dx^2}(0) = c_2, \frac{d^2y}{dx^2}(1) = d_2. \tag{4.2}$$

where $a(x)$ is singular and c_1, c_2, d_1, d_2 are real constants. Using (1.1), we write the problem (4.1) − (4.2) as follows:

$$\frac{d^2y}{dx^2}(x) = z(x), \tag{4.3}$$

$$\frac{d^2z}{dx^2}(x) = a(x)y(x) + d(x), \tag{4.4}$$

$$y(0) = c_1, y(1) = d_1, \tag{4.5}$$

$$z(0) = c_2, z(1) = d_2. \tag{4.6}$$

We use the method(3.14) to the BVP $(4.3)-(4.6)$. The method is given as follows:

$$\begin{aligned}
\sigma Y_{j-1}-Y_j(1+\sigma)+Y_{j+1} &= h_j^2\sigma(R\bar{Z}_{j+\frac{1}{2}}+Q\hat{Z}_j+P\bar{Z}_{j-\frac{1}{2}}), \qquad (4.7)\\
\sigma Z_{j-1}-Z_j(1+\sigma)+Z_{j+1} &= h_j^2\sigma(R(a_{j+\frac{1}{2}}\bar{Y}_{j+\frac{1}{2}}+d_{j+\frac{1}{2}})\\
&\quad +Q(a_j\hat{Y}_j+d_j)+P(a_{j-\frac{1}{2}}\bar{Y}_{j-\frac{1}{2}}+d_{j-\frac{1}{2}})). \qquad (4.8)
\end{aligned}$$

Then, we approximate $a_{j\pm\frac{1}{2}}$ for the BVP $(4.7)-(4.8)$ as

$$a_{j-\frac{1}{2}}=a_j-\frac{h_ja'_j}{2}+\frac{h_j^2a''_j}{8}+O(h_j^3), \qquad (4.9)$$

$$a_{j+\frac{1}{2}}=a_j+\frac{\sigma h_ja'_j}{2}+\frac{(h_j\sigma)^2a''_j}{8}+O(h_j^3). \qquad (4.10)$$

Similarily, we approximate $d_{j\pm\frac{1}{2}}$. Using the relations $(4.9)-(4.10)$ in $(4.7)-(4.8)$ we get,

$$\begin{aligned}
\sigma Y_{j-1}-Y_j(1+\sigma)+Y_{j+1} &= h_j^2\sigma(R\bar{Z}_{j+\frac{1}{2}}+Q\hat{Z}_j+P\bar{Z}_{j-\frac{1}{2}}), \qquad (4.11)\\
\sigma Z_{j-1}-Z_j(1+\sigma)+Z_{j+1} &= h_j^2\sigma(R(a_{j+\frac{1}{2}}\bar{Y}_{j+\frac{1}{2}}+d_{j+\frac{1}{2}})\\
&\quad +Q(a_j\hat{Y}_j+d_j)+P(a_{j-\frac{1}{2}}\bar{Y}_{j-\frac{1}{2}}+d_{j-\frac{1}{2}})). \qquad (4.12)
\end{aligned}$$

Finally, substituting $(3.1)-(3.12)$ in $(4.11)-(4.12)$ we get the difference equation of BVP $(4.3)-(4.6)$as follows:

$$\begin{bmatrix} b_j^{11} & b_j^{12} \\ b_j^{21} & b_j^{22} \end{bmatrix}\begin{bmatrix} Y_{j-1} \\ Z_{j-1} \end{bmatrix}+\begin{bmatrix} d_j^{11} & d_j^{12} \\ d_j^{21} & d_j^{22} \end{bmatrix}\begin{bmatrix} Y_j \\ Z_j \end{bmatrix}+\begin{bmatrix} p_j^{11} & p_j^{12} \\ p_j^{21} & p_j^{22} \end{bmatrix}\begin{bmatrix} Y_{j+1} \\ Z_{j+1} \end{bmatrix}=\begin{bmatrix} \psi_j^1 \\ \psi_j^2 \end{bmatrix}, \qquad (4.13)$$

where

$$\begin{aligned}
b_j^{11} &= -\sigma + \frac{h_j^4}{2}\sigma^2 Q\delta a_j, \quad b_j^{12} = \frac{h_j^2\sigma^2}{2}, \\
b_j^{21} &= \frac{h_j^2}{2}\sigma^2 R a_{j-\frac{1}{2}}, \qquad b_j^{22} = -\sigma + \frac{h_j^4}{2}\sigma^2 Q\delta a_j, \\
d_j^{11} &= (1+\sigma) + Q\delta a_j h_j^4\sigma^2, \qquad d_j^{12} = \frac{h_j^2\sigma^2(2Q+R+P)}{2}, \\
d_j^{21} &= \frac{\sigma^2}{2}[h_j^2 a_j(2Q+R+P) + h_j^3(-P+\sigma R)\frac{a_j'}{2} + R\frac{h_j^4\sigma^2 a_j''}{8}], \\
& \qquad\qquad diag_j^{22} = (1+\sigma) + Q\delta a_j h_j^4\sigma^2, \\
p_j^{11} &= -1 + \frac{a_j h_j^4\sigma^2 Q\delta}{2}, \quad p_j^{12} = \frac{Rh_j^2\sigma^2}{2}, \\
p_j^{21} &= h_j^2\sigma^2\frac{R}{2}a_{j+\frac{1}{2}}, \qquad p_j^{22} = -1 + \frac{a_j h_j^4\sigma^2 Q\delta}{2}, \\
\psi_j^1 &= -h_j^4 2b_j Q\sigma^2, \\
& \psi_j^2 = -\sigma^2[h_j^2 d_j(2Q+P+R) + \frac{d_j' h_j^3}{2}(-P+R) + \frac{d_j'' h_j^4}{8}(R+P)].
\end{aligned}$$

5 Convergence Analysis

We provide the convergence of method (3.14) for the coupled second order BVP (4.3) − (4.6). The convergence of scalar singular BVP has been already provided by Mohanty [23]. Now, once the condition (4.5) − (4.6) is substituted in the difference equation (4.13), it is written in matrix form as follows:

$$H\hat{Y} + \hat{\psi} = \begin{bmatrix} b_j & d_j & p_j \end{bmatrix} \begin{bmatrix} \hat{Y_{j-1}} \\ \hat{Y_j} \\ \hat{Y_{j+1}} \end{bmatrix} + \hat{\psi}_j = \hat{0}, \tag{5.1}$$

where b_j, p_j, d_j are block elements of order 2 in tridiagonal block matrix H.

$$\begin{aligned}
\hat{Y} &= [\hat{Y}_1, \hat{Y}_2, ..., \hat{Y}_j, ...\hat{Y}_{N-1}]^T, where \qquad \hat{Y}_j = [Y_j, Z_j]^T, \\
\hat{\psi} &= [\hat{\psi}_1 + b_1[c_1, c_2]^T, \hat{\psi}_2, ..., \hat{\psi}_j, ...\hat{\psi}_{N-1} + p_{N-1}[d_1, d_2]^T]^T, \ where \ \hat{\psi}_j = [\psi_j^1, \psi_j^2]^T,
\end{aligned}$$

$\hat{0}$ is a zero vector with $N-1$ components.

Let $[[y_1, z_1]^T, [y_2, z_2]^T ......, [y_j, z_j]^T, ...[y_{N-1}, z_{N-1}]^T]^T \cong \hat{y}$ be the exact solution satisfying

$$H\hat{y} + \hat{\psi} + \hat{T}_j^3 = 0, \tag{5.2}$$

where $\hat{T}_j^3$ is the truncation error, then the error vector E is given by $\hat{y} - \hat{Y}$. We get the error equation from(5.1)and(5.2), *i.e.*, $HE = \hat{T}_j^3$. (5.3)

For some $k_1, k_2 > 0$, let $|a_j| \leq k_1$ and $|a_j'| \leq k_2$. Using (4.13) and neglecting the higher order terms of h_j we get,

$$\|p_j\|_\infty \leq \max_{1\leq j\leq N-2} \begin{cases} 1 + \frac{h_j^2\sigma^2 P}{2}, \\ 1 + \frac{h_j^2\sigma^2 P}{2}(k_1 + \frac{h_j\sigma}{2}k_2), \end{cases} \tag{5.4}$$

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$$\|b_j\|_\infty \leq \max_{2\leq j\leq N-1} \begin{cases} \sigma + \frac{h_j^2\sigma^2 R}{2}, \\ \sigma + \frac{h_j^2\sigma^2 R}{2}(k_1 + \frac{h_j}{2}k_2). \end{cases} \tag{5.5}$$

We prove the irreducibility of H for sufficiently small h_j as well as $\|b_j\|_\infty \leq \sigma$ and $\|p_j\|_\infty \leq 1$ from $(5.4) - (5.5)$.
Let the sum of elements of j_{th} row of H be sum_j,

$$sum_j = \begin{cases} \sigma + \frac{h_j^2\sigma^2}{12}(P + 2(R+Q)), j = 1, \\ \sigma + \frac{h_j^2\sigma^2 a_j'}{24}(R + 2(P+Q))a_j + \frac{\sigma^2}{2}(h_j^2 R a_j + h_j^3(-P + 2R\sigma)),\ j = 2, \end{cases} \tag{5.6}$$

$$sum_j = \begin{cases} \frac{h_j^2\sigma^2}{2}(R + Q + P),\ j = 3(2)N-4, \\ \frac{h_j^2\sigma^2 a_j}{2}(R+Q+P) + \frac{h_j^3\sigma^2 a_j'}{4}(-2P + \sigma R),\ j = 4(2)N-3, \end{cases} \tag{5.7}$$

$$sum_j = \begin{cases} 1 + \frac{h_j^2\sigma^2}{12}(R + 2(Q+P)),\ j = N-2, \\ 1 + \frac{h_j^2\sigma^2 a_j}{12}(R + 2(Q+P)) + \frac{h_j^3\sigma^2 a_j'}{4}(-2R + P\sigma),\ j = N-1. \end{cases} \tag{5.8}$$

We can easily prove that H is Monotone using $0 < L \leq min(L_1, L_2)$ in $(5.6) - (5.8)$ and for sufficiently small h_j. Therefore, $H^{-1} \geq 0$ and exist. Hence by (5.3) we have,

$$||E|| = ||H^{-1}||||\hat{T}_j^3||. \tag{5.9}$$

Now for sufficiently small h_j, by $(2.23) - (2.25)$ and $(5.6) - (5.8)$ we can say that:

$$sum_j > \begin{cases} \frac{h_j^2\sigma(2+3\sigma)}{12}, & j = 1, \\ \frac{h_j^2\sigma(2+3\sigma)L}{12}, & j = 2, \end{cases} \tag{5.10}$$

$$sum_j \geq \begin{cases} \frac{h_j^2(\sigma+1)}{2}, & j = 3(2)N-4, \\ \frac{h_j^2(\sigma+1)L}{2}, & j = 4(2)N-3, \end{cases} \tag{5.11}$$

$$sum_j > \begin{cases} \frac{h_j^2\sigma(2\sigma+3)}{12}, & j = N-2, \\ \frac{h_j^2\sigma(2\sigma+3)L}{12}, & j = N-1. \end{cases} \tag{5.12}$$

We can also say for $\sigma \neq 0$:

$$\begin{aligned} sum_j &> \max[\frac{h_j^2\sigma(2+3\sigma)}{12}, \frac{h_j^2\sigma(2+3\sigma)L}{12}] \\ &= \frac{h_j^2\sigma(2+3\sigma)L}{12}, \text{for} \quad j = 1, 2, \end{aligned} \tag{5.13}$$

$$\begin{aligned} sum_j &\geq \max[\frac{h_j^2(1+\sigma)}{2}, \frac{h_j^2(1+\sigma)L}{2}] \\ &= \frac{h_j^2(1+\sigma)L}{2}, \quad \text{for } j = 3(1)N-3, \end{aligned} \tag{5.14}$$

$$\begin{aligned} sum_j &> \max[\frac{h_j^2\sigma(2\sigma+3)}{12}, \frac{h_j^2\sigma(2\sigma+3)L}{12}] \\ &= \frac{h_j^2\sigma(2\sigma+3)L}{12}, \text{for} \quad j = N-2, N-1. \end{aligned} \tag{5.15}$$

Then, we use a result proved by Varga [33] for $i = 1(1)N-1$,

$$H_{i,j}{}^{-1} \leq \frac{1}{sum_j}, \quad \text{where } H_{i,j}{}^{-1} \text{ is the } (i,j)^{th} \text{element of } H^{-1} \tag{5.16}$$

By using $(5.13)-(5.15)$, we have

$$\frac{1}{sum_j} \leq \begin{cases} \frac{12}{h_j^2(3\sigma+2)\sigma L}, & j = 1,2, \\ \frac{2}{h_j^2(\sigma+1)L}, & j = 3(1)N-3, \\ \frac{12}{h_j^2(2\sigma+3)\sigma L}, & j = N-2, N-1. \end{cases} \tag{5.17}$$

Now, we show that the error defined in equation (5.9) is bounded and is of order $O(h_j^3)$. For this, we define norm of H^{-1} and $\hat{T}_j^3$ such that,

$$\| H_{j,i}{}^{-1} \| = \max_{j\in[1,N-1]} \sum_{i=1}^{N-1} | H_{j,i}{}^{-1} |, also \| T \| = \max_{j\in[1,N-1]} | \hat{T}_j^3 | . \tag{5.18}$$

Thus, using (5.3) and $(5.16)-(5.18)$ we get the bound for the error term as follows:

$$\| E \| \leq O(h_j^5) \frac{12}{h_j^2 L\sigma} \frac{(6\sigma^3 + 18\sigma^2 + 16\sigma + 5)}{(6\sigma^3 + 19\sigma^2 + 19\sigma + 6)} = O(h_j^3). \tag{5.19}$$

This proves the method (3.14) has third order convergence for BVPs $(4.1)-(4.2)$. Therefore, we can say that method (3.14) has third order convergence for BVP $(1.1)-(1.2)$. Similarly, method (3.13) has second order convergence.

6 Numerical Illustrations

We have solved seven problems. For quasi-variable mesh and uniform mesh, we have tabulated root mean square errors and maximum absolute errors respectively in Tables 1-7. We have chosen $h_1 = \frac{(1-\sigma)}{(1-\sigma^N)}, \sigma \neq 1$. The remaining h_j 's are calculated by the relation $h_j = \sigma h_{j-1}, j = 2(1)N-1$. Figures 1-7 presents the graphs of numerical solution and the exact solution in case of fourth order method based on uniform mesh. Related numerical results are provided in Table 1-7.

Gauss Elimination and Newton's method for block elements has been used for solving system of linear and nonlinear BVPs respectively with initial approximation $y_0 = 0$. The order of convergence (OC) for fourth order method based on uniform mesh is also provided. Matlab 07 has been used for doing all calculations.

Problem 6.1 (Nonlinear boundary value problem)

$$\frac{d^4y(x)}{dx^4} = 6e^{-4\ y} - \frac{12}{(1+x)^4},$$
$$y(0) = 0, y(1) = .6931, \frac{d^2y}{dx^2}(0) = -1, \frac{d^2y}{dx^2}(1) = -.25.$$

Table 1: Problem 6.1

	Off-step mesh			
N	*Method*1	*Method*2	*Uniform mesh method*	[29]
8	4.4398e-003	4.8610e-006	7.2499e-007	0.37e-005
16	2.0758e-003	1.2961e-006	4.6937e-008	0.29e-006
32	1.3702e-003	6.7628e-007	2.9600e-009	0.19e-007

The exact solution is given by $y(x) = \log(1 + x)$. In Table 1, results for quasi-variable mesh taking $\sigma = 0.9$ and for uniform mesh is tabulated.

Problem 6.2 (Sixth order linear boundary value problem):

$$\frac{d^6y(x)}{dx^6} + y(x) = 6(5\sin(x) + 2x\cos(x)), x \in [0, 1]$$
$$y(0) = 0, \frac{d^2y}{dx^2}(0) = 0, \frac{d^4y}{dx^4}(0) = 0,$$
$$y(1) = 0, \frac{d^2y}{dx^2}(1) = 3.84416, \frac{d^4y}{dx^4}(1) = -14.42007.$$

The exact solution is $y(x) = (x^2 - 1)\sin(x)$. In Table 2, results for quasi-variable mesh taking $\sigma = 0.9$ and for uniform mesh is tabulated.

Problem 6.3 (Fourth order non linear boundary value problem)

$$\frac{d^4y(x)}{dx^4} = 3(\frac{dy}{dx})^2 + 4.5y^3, x \in [0, 1]$$
$$y(0) = 4, \frac{d^2y}{dx^2}(0) = 24, y(1) = 1, \frac{d^2y}{dx^2}(1) = 1.5e.$$

The exact solution is $y(x) = \frac{4}{(1+2x+x^2)}$. In Table 3, results for quasi-variable mesh taking $\sigma = 0.9$ and for uniform mesh is tabulated.

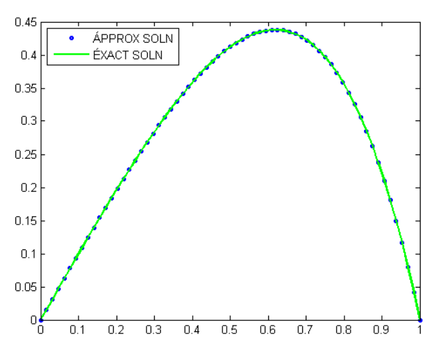

Figure 1: Exact solution vs Numerical solution in uniform mesh method

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Table 2: *Problem 6.2*

	Off-step mesh				
N	*Method*1	*Method*2	*Uniform mesh method*	[3]	[26]
8	6.4952e-004	4.2946e-006	6.5901e-007	1.5379 e-006	8.1514e-005
16	5.9397e-004	9.9183e-007	4.1831e-008	1.9790 e-007	2.1052 e-005
32	5.1433e-004	4.7874e-007	2.6133e-009	4.0596 e-008	5.3084 e-006

Table 3: *Problem 6.3*

	Off-step mesh			
N	*Method*1	*Method*2	*Uniform mesh method*	[25]
8	1.2451e-003	8.4710e-005	2.2780e-005	1.44 e-003
16	2.7555e-004	2.1499e-005	1.5362e-006	9.33 e-004
32	4.0919e-004	9.7519e-006	9.8628e-008	5.90 e-005
64	3.1887e-004	6.4390e-006	6.2300e-009	3.69 e-006

Problem 6.4 (Sixth order non linear boundary value problem)

$$\begin{aligned}
&\frac{d^6y(x)}{dx^6} = y^2e^{-x},\ x \in [0,1]\\
&y(0) = 1, y(1) = e,\\
&\frac{d^2y}{dx^2}(0) = 1, \frac{d^2y}{dx^2}(1) = e,\\
&\frac{d^4y}{dx^4}(0) = 1, \frac{d^4y}{dx^4}(1) = e.
\end{aligned}$$

The exact solution is $y(x) = e^x$. In Table 4, results for quasi-variable mesh taking $\sigma = 0.9$ and for uniform mesh is tabulated.

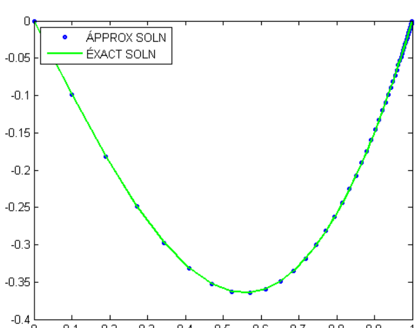

Figure 2: Exact solution vs Numerical solution in uniform mesh method

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Table 4: *Problem 6.4*

	Off-step mesh			
N	$Method1$	$Method2$	$Uniform\ mesh\ method$	[14]
8	2.64952e-004	2.0457e-007	5.1651e-008	7.02e-006
16	5.9397e-004	4.9805e-008	3.2495e-009	4.35e-006
32	5.1433e-004	2.4007e-008	2.0334e-010	7.87e-007

Problem 6.5(Fourth order non-linear singular boundary value problem)

$$x\frac{d^4y(x)}{dx^4} + \frac{4d^3y(x)}{dx^3} = xy^2 - 4\cos(x) - xsin(x), x \neq 0.$$

The exact solution is $y(x) = \sin(x)$. In Table 5, results for quasi-variable mesh taking $\sigma = 0.9$ and for uniform mesh is tabulated.

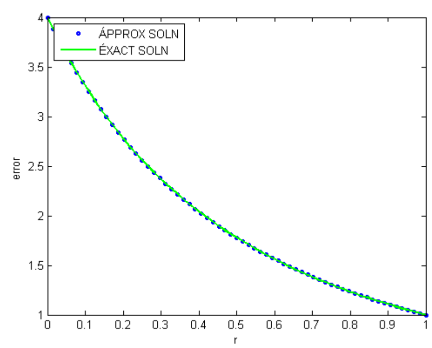

Figure 3: Exact solution vs Numerical solution in uniform mesh method

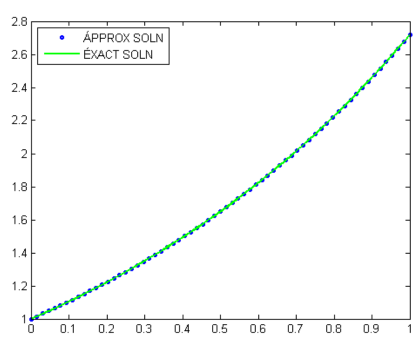

Figure 4: Exact solution vs Numerical solution in uniform mesh method

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Table 5: *Problem 6.5*

	Off-step mesh				
N	$Method1$	$Method2$		$Uniform\ mesh\ method$	OC
8	1.4374e-004	4.2067e-006		1.6791e-006	-
16	8.8426e-005	1.2629e-006		1.5413e-007	3.4455
32	5.7765e-005	6.0728e-007		1.2889e-008	3.5810
64	4.0494e-005	4.0146e-007		7.7938e-010	4.1476

Problem 6.6 (Sixth order non-linear singular boundary value problem)

$$x\frac{d^6y(x)}{dx^6} + 6\frac{d^5y(x)}{dx^5} + 2xy(x) = xe^y, x \neq 0.$$

The exact solution is $y(x) = e^x$. In Table 6, results for quasi-variable mesh taking $\sigma = 0.9$ and for uniform mesh is tabulated.

Table 6: *Problem 6.6*

	Off-step mesh				
N	$Method1$	$Method2$		$Uniform\ mesh\ method$	OC
8	6.1864e-004	5.6862e-007		4.5498e-007	-
16	3.1623e-004	1.4817e-007		3.7876e-008	3.5865
32	2.2083e-004	7.6631e-008		2.9520e-009	3.6815
64	2.0493e-004	6.7278e-008		2.2125e-010	3.7379

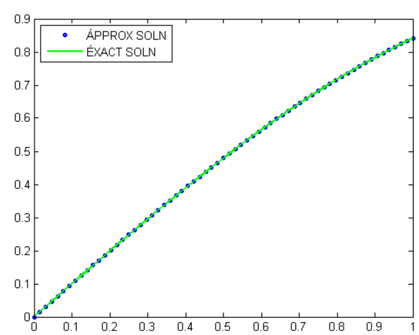

Figure 5: Exact solution vs Numerical solution in uniform mesh method

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Problem 6.7 (System of second order boundary value problem)

$$\begin{aligned}
&\frac{d^2y(x)}{dx^2} + \frac{dy(x)}{dx} + xy(x) + \frac{dz(x)}{dx} + 2xz(x) = g_1(x),\\
&\frac{d^2z(x)}{dx^2} + z(x) + 2\frac{dy(x)}{dx} + x^2y(x) = g_2(x),\\
&y(0) = 0, z(0) = 1, y(1) = 0, z(1) = 1,
\end{aligned}$$

where $g_1(x) = -2\cos(x)(1+x) + \pi cos(x\pi) + 2xsin(x\pi) + 2\sin(x)(2x-2-x^2)$, $g_2(x) = -4\cos(x)(x-1) + 2\sin(x)(2-x^2+x^3) + (1-\pi^2)\sin(x\pi)$ and $x \in [0,1]$. The exact solution is $y(x) = 2(1-x)\sin(x), z(x) = \sin(x\pi)$.

Table 7: *Problem 6.7*

	z		y	
N	[8]	*Uniform mesh method*	[8]	*Uniform mesh method*
.08	7.5e-004	3.5686e-007	2.2e-004	1.8284e-006
.24	8.2e-004	1.4754e-006	2.3e-004	2.0723e-006
.40	6.5e-004	2.5123e-006	2.3e-004	6.2430e-007
.56	2.8e-004	3.1366e-006	2.2e-004	3.9577e-006
.72	2.6e-004	2.9899e-006	2.6e-004	5.6498e-006
.88	8.0e-004	1.2382e-005	5.5e-004	3.9716e-006
.96	4.8e-004	1.4964e-006	3.1e-004	1.5857e-006

7 Final Remarks

In this paper, two methods of second and third order respectively have been developed to solve singular BVPs both linear as well as nonlinear. For numerical illustration, we have considered seven problems consisting of fourth and sixth order linear and nonlinear BVPs. Table $1-4, 7$ proves improvement in results when compared with problems

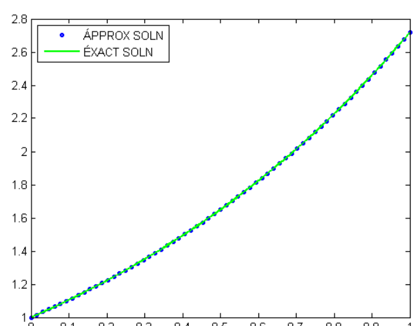

Figure 6: Exact solution vs Numerical solution in uniform mesh method

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solved by methods using extrapolation, polynomial and non polynomial splines and also by using reproducing kernel space method.

In our methods minimal grid points i.e., three grid points at a time has been used as compared to existing methods. Due to the use of three grid points, the numerical scheme is converted to a tri-diagonal representation of system of difference equations which can be easily solved by any standard method available in the literature. Also, due to the use of off-step mesh, singularity has been controlled in singular BVPs. We have also solved nonlinear singular BVP and so far such kind of BVP has not been solved. Therefore, for such problems we have presented the numerical order of convergence(OC) based on uniform mesh.

The methods developed are effective and straight forward and can be extended to solve boundary value problems with cartesian as well as polar coordinates. Due to the ability to operate with polar coordinate, many problems on fluid flow with polar symmetry can be attended. Moreover, we can also use the methods to solve wide variety of higher order singularly perturbed BVPs.

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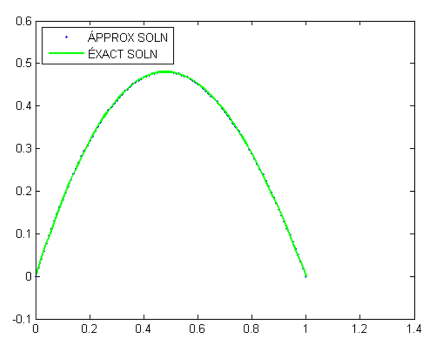

Figure 7: Exact solution vs Numerical solution in uniform mesh method

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Numerical study of the space fractional Burger's equation by using Lax-Friedrichs-implicit scheme

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Abstract

This paper deals with the numerical solution of space fractional Burger's equation using the implicit finite difference scheme and Lax-Friedrichs-implicit finite difference scheme respectively. The Riemann-Liouville based fractional derivative (non-integer order) is fitted for the diffusion term of fractional order $1.0 < \alpha \leq 2.0$. The Mathematical induction is used to estimate a stability of both the implicit and Lax-Friedrichs-implicit schemes. The study shows that the implicit based scheme is stable and the results are good in agreement with the exact solution. Finally, the significance of space fractional order with respect to the solution is discussed. It is noted that the solution of space fractional Burger's equation get affected by changing the space fractional order.

Key words: Lax-Friedrichs, implicit scheme, fractional calculus, finite difference method

1

1 Introduction

Fractional Calculus plays an important role in various fields of science and engineering. Examples include ground water flow modeling, electric circuit design, quantum mechanics, optics, plasma model, dengue fever transmission dynamics and atmospheric CO_2 dynamics model [1, 2, 3, 4, 5]. Due to its wide applications, solving techniques of those fractional equations are extensively improved by the researchers. For instance, Goswami et al. [6] used Homotopy perturbation Sumudu transform for solving time-fractional regularized long wave equations. Later, they used the techniques to find the solutions for the time fractional Schrdinger equations and fractional equal width equations (describes the

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hydro-magnetic waves) [7, 8]. Also, Goswami et al. [9] made a mixed approach of Homotopy perturbation and Laplace transform to solve the fifth order KdV equations in order to illustrate the plasma's magneto-acoustic waves. Recently, Hashmi et al. [10] used B-spline method to solve the fractional telegraph equation and quoted that the scheme is efficient. In numerical methods there are numerous methods including finite difference method (FDM), finite element method, finite volume method etc. Out of this methods FDM is a pioneering tool used among the investigators. In the present investigation, we establish two schemes namely the implicit FDM and Lax-Friedrichs implicit FDM for the space fractional Burger's equation (SFBE).

Fractional calculus application gives a real system better than integer-order. The Burger's equations arises in various domain such as fluid and gas dynamics, theory of shock waves, traffic flow, etc [11, 12, 13, 14]. Many researchers have applied various analytical techniques, numerical algorithms/schemes for extracting the solution for the Burger's equation. The exact solution and explicit FDM solutions for the 1-D Burger's equation was surveyed by Kutluay et al. [15]. Aksan and Ozdes [16] constructed variational method for solving the Burger's equation. Inan and Bahadir [17] converted non-linear Burger's equation into linear using Hopf-Cole transformation and obtained Numerical solution (NS) using explicit exponential FDM. Pandey et al. [18] coupled Hopf-Cole transformation and Douglas FDM to get the NS with accuracy of second order in time and fourth order in space.

Zhang et al.[19] used the implicit FDM to solve the fractional convection-diffusion equation. It is found that the NS is unconditionally stable. Sousa [20] obtained the NS for the fractional advection diffusion equation using explicit-central difference FDM, explicit-upwind FDM and Lax-Wendroff FDM. The study consider Riemann-Liouville fractional derivative for space fractional and Caputo fractional derivative for the time derivative. The result shows that all the explicit FDM schemes are stable under restricted conditions. Later, Sousa [21] presented the explicit-Lax-Wendroff method for the Riemann-Liouville derivative based space fractional advection diffusion equation. The study illustrates that the scheme is second order accurate and conditionally stable. Bekir and Gnerb [22] and Das et al. [23] used (G'/G) expansion method to solve the modified Riemann-Liouville derivative based fractional Burger's equation. Esen and Tasbozan[24] solved the time fractional Burger's equation by applying the B-spline quadratic Galerkin method. Moreover, Esen and Tasbozan[25] used finite element method based cubic B-spline for the time fractional Burger's equation. They also compared the NS with the various exact solutions (ES) and found that the scheme is stable and accurate. Rawashdeh [26] proposed a new scheme named the fractional reduced differential transform to solve the TFBE. It is noted that the proposed scheme is accurate and good comparable with the ES. Yokus [27] studied the FDM based NS with respect to the fractional derivatives such as Caputo, shifted Grunwald and Riemann-Liouville and obtained the solutions using the software Mathematica 11. Saad and Eman[28] have applied the variational iteration method (VIM) for the Riemann-Liouville based fractional Burger's equation and compared the results with the ES.

In this work, we propose a numerical solution based on implicit FDM scheme and Lax-Friedrichs FDM scheme to solve a non-linear SFBE. Generally, the Lax-Friedrichs method is used for achieving the solutions for a hyperbolic based PDE's [29]. In general, an implicit scheme is the most well-known schemes for approximating the PDEs. This paper presents an approximation based on Lax-Friedrichs-implicit FDM to non-linear SFBE with appropriate initial/boundary conditions. The stability of a proposed scheme is analysed along with the numerical results.

2 Mathematical equation

Time fractional Burgers' equation was discussed in the articles [16, 24, 25, 26]. Following their study, we consider the non-linear SFBE as,

$$\frac{\partial u(x,t)}{\partial t} + u\frac{\partial u(x,t)}{\partial x} = \mu\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}}, (x,t) \in [a,b] \times (0, T_{max}] \tag{2.1}$$

included with initial values

$$u(x,0) = u_0(x) \tag{2.2}$$

and respective boundary values

$$u(0,t) = h_1(t); u(1,t) = h_2(t), t \in [0,T] \tag{2.3}$$

where $\mu > 0$ is kinematic viscosity, $u_0(x)$, $h_1(t)$ and $h_2(t)$ are specified boundaries. $u(x)$ is unknown functional.

To solve the SFBE in this work, let us consider the Riemann-Liouville fractional derivatives [20, 21, 30].

$$({}_0D_x^{\alpha})u(x,t) = \frac{1}{\Gamma(r-\alpha)}\frac{d^r}{dx^r}\int_L^x \frac{u(t)}{(x-t)^{\alpha-r+1}}dt, \alpha > 0 \tag{2.4}$$

where $\Gamma\left(.\right)$ is the Gamma function.

For space fractional derivative $({}_0D_x^{\alpha})u\left(x,t\right)$, we taken the Grunwald and shifted-Grunwald formula at level t_{n+1} [31].

$$\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} = \frac{1}{h^{\alpha}}\sum_{j=0}^{i+1} g_j^{\alpha}\; u_{i-j+1}^{k+1} + O\left(h\right) \tag{2.5}$$

where $g_j^{\alpha} = \dfrac{\alpha\left(\alpha-1\right)\ldots.(\alpha-k+1)}{j!}$,

We can express, $\quad g_0^{\alpha} = 1, \ldots,\; g_j^{\alpha} = \left(1 - \dfrac{\alpha}{j}\right) g_{j-1}^{\alpha},\; j = 1, 2, 3, \ldots.$

3

2.1 Implicit scheme

The implicit scheme is one of the more accurate scheme for a non-linear Burger's equation [32]. Here, we consider a same for SFBE due to its stability than the explicit scheme [21, 31, 33, 34].

Let $u(x_i, t_k)$ is denoted as u_i^k. Define, $t_k = k\tau, \quad k = 0, 1, 2, ..., n;\ x_i = ih, \quad i = 0, 1, 2, ..., m$. Here, $h = L/m$ is the step size on space and $\tau = T/n$ is the step size on time respectively. Now,let us consider the nonlinear term, $u^{k+1}u_x^{k+1}$ by denoting it on Taylor expansion using the explicit time layer. We approximate the equation (2.1) by using an implicit FDM and approximated Riemann-Liouville derivatives equation (2.5) in space fractional viscous terms as follows.

$$
\begin{aligned}
\frac{(u_i^{k+1} - u_i^k)}{\tau} + \frac{u_i^k}{2}\left(\frac{u_{i+1}^{k+1} - u_{i-1}^{k+1}}{2h}\right) + \frac{u_i^{k+1}}{2}\left(\frac{u_{i+1}^{k} - u_{i-1}^{k}}{2h}\right) \\
= \frac{\mu}{h^\alpha}\sum_{j=0}^{i+1} g_j^\alpha u_{i-j+1}^{k+1}
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
\left(u_i^{k+1} - u_i^k\right) + \tau\frac{u_i^k}{2}\left(\frac{u_{i+1}^{k+1} - u_{i-1}^{k+1}}{2h}\right) + \tau u_i^{k+1}\left(\frac{u_{i+1}^{k} - u_{i-1}^{k}}{4h}\right) \\
= \tau\frac{\mu}{h^\alpha}\sum_{j=0}^{i+1} g_j^\alpha u_{i-j+1}^{k+1}
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
-\tau\left(\frac{u_i^k}{4h} + \frac{\mu}{h^\alpha}g_2^\alpha\right)u_{i-1}^{k+1} + \left(1 + \frac{\tau\left(u_{i+1}^{k} - u_{i-1}^{k}\right)}{4h} - \frac{\mu\tau}{h^\alpha}g_1^\alpha\right)u_i^{k+1} \\
+\tau\left(\frac{u_i^k}{4h} - \frac{\mu}{h^\alpha}g_0^\alpha\right)u_{i+1}^{k+1} - \tau\frac{\mu}{h^\alpha}\sum_{j=3}^{i+1} g_j^\alpha u_{i-j+1}^{k+1} = u_i^k
\end{aligned}
\tag{2.8}
$$

When $k = 0$,

$$
\begin{aligned}
-\tau\left(\frac{u_i^0}{4h} + \frac{\mu}{h^\alpha}g_2^\alpha\right)u_{i-1}^{1} + \left(1 + \frac{\tau\left(u_{i+1}^{0} - u_{i-1}^{0}\right)}{4h} - \frac{\mu\tau}{h^\alpha}g_1^\alpha\right)u_i^{1} \\
+\tau\left(\frac{u_i^0}{4h} - \frac{\mu}{h^\alpha}g_0^\alpha\right)u_{i+1}^{1} - \tau\frac{\mu}{h^\alpha}\sum_{j=3}^{i+1} g_j^\alpha u_{i-j+1}^{1} = u_i^0
\end{aligned}
\tag{2.9}
$$

When $k \geq 1$,

$$
\begin{aligned}
-\tau\left(\frac{u_i^k}{4h} + \frac{\mu}{h^\alpha}g_2^\alpha\right)u_{i-1}^{k+1} + \left(1 + \frac{\tau\left(u_{i+1}^{k} - u_{i-1}^{k}\right)}{4h} - \frac{\mu\tau}{h^\alpha}g_1^\alpha\right)u_i^{k+1} \\
+\tau\left(\frac{u_i^k}{4h} - \frac{\mu}{h^\alpha}g_0^\alpha\right)u_{i+1}^{k+1} - \tau\frac{\mu}{h^\alpha}\sum_{j=3}^{i+1} g_j^\alpha u_{i-j+1}^{k+1} = u_i^k
\end{aligned}
\tag{2.10}
$$

Rewriting above equation, we get

$$a_i^k u_{i-1}^{k+1} + b_i^k u_i^{k+1} + c_i^k u_{i+1}^{k+1} = u_i^k + d_i^k \tag{2.11}$$

where, $a_i^k = -\tau\left(\frac{u_i^k}{4h} + \frac{\mu}{h^\alpha} g_2^\alpha\right)$, $b_i^k = \left(1 + \frac{\tau\left(u_{i+1}^k - u_{i\ -1}^k\right)}{4h} - \frac{\mu\tau}{h^\alpha} g_1^\alpha\right)$, $c_i^k = \tau\left(\frac{u_i^k}{4h} - \frac{\mu}{h^\alpha} g_0^\alpha\right)$, $d_i^k = \tau\frac{\mu}{h^\alpha}\sum_{j=3}^{i+1} g_j^\alpha u_{i-j+1}^{k+1}$

The boundary/initial conditions are,

$$u_i^0 = u\left(ih\right), \quad u_0^k = h_1\left(t\right), \quad u_m^k = h_2\left(t\right)$$

where $k = 0,\ 1,\ 2, ..., n,$ $i = 0,\ 1,\ 2,\ ...., m.$ The truncation error is $O\left(\tau^2, h^2\right).$

2.1.1 Stability analysis - implicit FDM

Let us investigate the stability of the numerical implicit scheme (2.8) by using von-Neumann analysis. Let U_j^k is the ES of $u(x,t)$ at the point (x_j, t_k). Define

$$e_j^k = U_j^k - u_j^k \tag{2.12}$$

Then, by substituting Equation (2.12) into Equation (2.11),we have

$$a_i^k e_{i-1}^{k+1} + b_i^k e_i^{k+1} + c_i^k e_{i+1}^{k+1} = e_i^k + d_i^k \tag{2.13}$$

We put $e_i^k = \rho^k\ e^{ipjh}$ $(i = \sqrt{-1})$, in equation (2.6) and p is the wave number.

$$\rho^{k+1}\left[\tau\left(\frac{\rho^k\ e^{ipjh}}{2h}\right) isin\left(ph\right) + \left(1 + \frac{\tau\left(\ isin\left(ph\right)\rho^k\ \right)}{2h}\right) - \tau\frac{\mu}{h^\alpha}\sum_{r=0}^{i+1} g_j^\alpha\ e^{ip(1-r)h}\right] = \rho^k \tag{2.14}$$

$$\frac{\rho^{k+1}}{\rho^k} = \frac{1}{\left[\tau\left(\frac{\rho^k\ e^{ipjh}}{2h}\right) isin\left(ph\right) + \left(1 + \frac{\tau\left(\ isin\left(ph\right)\rho^k\ e^{ipjh}\ \right)}{2h}\right) - \tau\frac{\mu}{h^\alpha}\sum_{r=0}^{i+1} g_j^\alpha\ e^{ip(1-r)h}\right]} \leq 1 \tag{2.15}$$

It is obvious that the above scheme is unconditionally stable.

2.2 Lax-Friedrichs Scheme

As a result of its application to a nonlinear space fractional problem and the dissipative nature of the solution, the Lax-Friedrichs scheme is considered to be a classic first-order method. The Lax-Friedrichs scheme of the fractional equation (2.1) is approximated by as below:

$$\left(\frac{u_i^{k+1}-\frac{1}{2}\left(u_{i-1}^k+u_{i+1}^k\right)}{\tau}\right)+\frac{u_i^k}{2}\left(\frac{u_{i+1}^{k+1}-u_{i-1}^{k+1}}{2h}\right)+\frac{u_i^{k+1}}{2}\left(\frac{u_{i+1}^k-u_{i-1}^k}{2h}\right) = \frac{\mu}{h^\alpha}\sum_{j=0}^{i+1} g_j^\alpha u_{i-j+1}^{k+1} \tag{2.16}$$

$$-\tau(\frac{u^k}{4h})u_{i-1}^{k+1}+(1+\frac{\tau(u_{i+1}^k-u_{i-1}^k)}{4h})u_i^{k+1}+\tau\left(\frac{u^k}{4h}\right)u_{i+1}^{k+1}-\tau\frac{\mu}{h^\alpha}\sum_{j=0}^{i+1} g_j^\alpha u_{i-j+1}^{k+1} = \frac{1}{2}\left(u_{i-1}^k+u_{i+1}^k\right) \tag{2.17}$$

When $k=0$

$$-\tau(\frac{u^0}{4h})u_{i-1}^1+(1+\frac{\tau(u_{i+1}^0-u_{i-1}^0)}{4h})u_i^1+\tau(\frac{u^0}{4h})u_{i+1}^1-\tau\frac{\mu}{h^\alpha}\sum_{j=0}^{i+1} g_j^\alpha u_{i-j+1}^1 = \frac{1}{2}(u_{i-1}^0+u_{i+1}^0) \tag{2.18}$$

When $k\geq 1$

$$-\tau(\frac{u^k}{4h})u_{i-1}^{k+1}+(1+\frac{\tau(u_{i+1}^k-u_{i-1}^k)}{4h})u_i^{k+1}+\tau(\frac{u^k}{4h})u_{i+1}^{k+1}-\tau\frac{\mu}{h^\alpha}\sum_{j=0}^{i+1} g_j^\alpha u_{i-j+1}^{k+1} = \frac{1}{2}(u_{i-1}^k+u_{i+1}^k) \tag{2.19}$$

2.2.1 Stability Analysis - Lax-Friedrichs-implicit FDM

Let us consider the von-Neumann based method in preparation for estimating the stability of Lax-Friedrichs implicit scheme for SFBE. Let U_i^k be the approximate solution of fractional schemes (2.17).

$$e_i^k = U_i^k - u_i^k \tag{2.20}$$

Define, $e_i^k = \rho^k e^{ipjh}(i = \sqrt{(-1)}$ in Eq. (2.17), We get

$$\rho^{k+1}\left[\tau\left(\frac{\rho^k e^{ipjh}}{2h}\right) isin\,(ph) + \left(1+\frac{\tau\,(\ isin\,(ph)\,e^{ipjh}\ \rho^k)}{2h}\right) - \tau\frac{\mu}{h^\alpha}\sum_{r=0}^{i+1} g_j^\alpha\ e^{ip(1-r)h}\right] = \left(\rho^k\ cos(ph)\right) \tag{2.21}$$

$$\frac{\rho^{k+1}}{\rho^k} = \frac{cos(ph)}{\left[\left(1+\frac{\tau\,(\ isin\,(ph)\,e^{ipjh}\ \rho^k)}{2h}\right) + \tau\left(\frac{\rho^k e^{ipjh}}{2h}\right) isin\,(ph)\ - \tau\frac{\mu}{h^\alpha}\sum_{r=0}^{i+1} g_j^\alpha\ e^{ip(1-r)h}\right]} \tag{2.22}$$

We know, the value of the $sin\,(ph)\ \ and\ cos\,(ph)\ \ \leq 1$

3 Numerical Results

The verification of NS and accuracy of the schemes (implicit FDM and Lax-Friedrichs-implicit FDM) are illustrated in this section. In addition, the behavior of the solution with respect to change in the parameters are considered. This types of Burger's equation are used in predicting the important real world applications such as fluid flow, contaminant flow, boundary layer flow, aquifer flow, etc.

The accuracy of the FDM based schemes are measured using the L_∞ error norm, which is defined below:

$$L_\infty = \parallel U_k - u_N \parallel_\infty = Max_j |U_k - (u_N)_j| \tag{3.1}$$

where U_k and u_N denotes the ES and NS respectively at the node points x_k, for some fixed time.

3.1 Example 1

Consider the space fractional Burger's equation with source term to find error values as follows:

$$\frac{\partial u(x,t)}{\partial t} + u\frac{\partial u(x,t)}{\partial x} = \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + f(x,t), (x,t) \in [a,b] \times (0, T_{max}] \tag{3.2}$$

with initial and boundary conditions as

$$u\,(x,0) = x; \qquad and \quad u\,(0,t) = 0; \qquad u\,(1,t) = \frac{1}{1+t}$$

The exact solution [27]

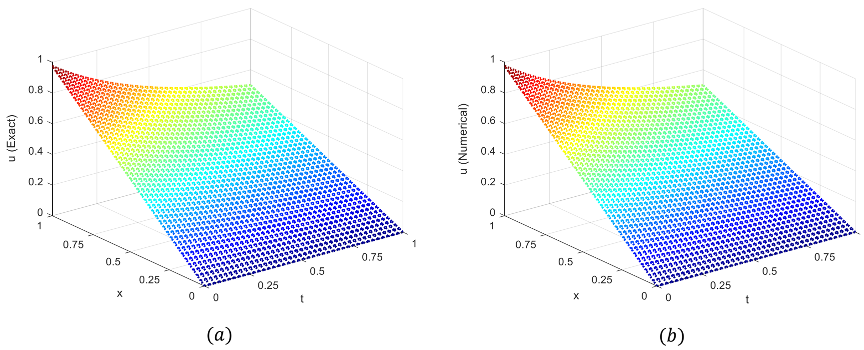


Figure 1: Comparison of (a). ES and (b). NS

$$u(x,t) = \frac{x}{1+t} \tag{3.3}$$

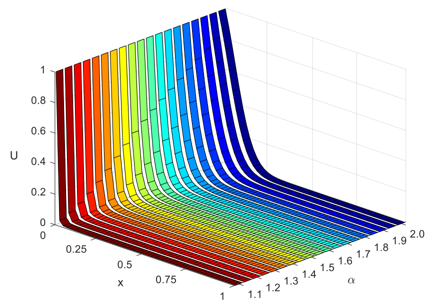
and the respective source term is,

$$f(x,t) = -\frac{1}{1+t}.\frac{1}{\Gamma(2-\alpha)}\mu x^{1-\alpha} \tag{3.4}$$

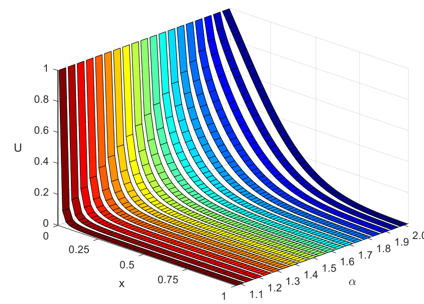
Table 1: Comparison the Maximum errors (L_∞) between ES and NS

τ	α	implicit FDM	Lax-Friedrichs FDM
1/100	1.9	$6.68689189E-03$	$4.27211449E-03$
1/100	1.7	$1.03293294E-02$	$9.42116044E-03$
1/100	1.5	$1.12734595E-02$	$1.07855788E-02$
1/100	1.3	$1.18441200E-02$	$1.14968475E-02$
1/100	1.1	$1.22581907E-02$	$1.18474280E-02$
1/1000	1.9	$3.30544871E-03$	$1.65415841E-03$
1/1000	1.7	$9.85641548E-03$	$7.98325112E-03$
1/1000	1.5	$9.89541454E-03$	$9.75634282E-03$
1/1000	1.3	$9.90254650E-03$	$9.76487132E-03$
1/1000	1.1	$9.91051481E-03$	$9.80037527E-03$

The verification of NS for the SFBE (2.18) with the ES is illustrated in the Fig. 1. The comparison is done against the time, $t = 0$ to 1 and space, $x = 0$ to 1. Both the NS and ES are good in comparable. Also, the Table. 3.1 shows the L_2 between the ES and NS. It is found that the Lax-Friedrichs-implicit FDM has lesser L_2 than the implicit FDM for every α.



(a)



(b)

Figure 2: variation of U at (a) $\mu = 0.1$ (b) $\mu = 1.0$

3.2 Example 2

Also, consider the SFBE without source term to find the characteristics of NS as

$$\frac{\partial U(x,t)}{\partial t} + u\frac{\partial U(x,t)}{\partial x} = \mu\frac{\partial^{\alpha} U(x,t)}{\partial x^{\alpha}} \tag{3.5}$$

with initial and boundary conditions as

$$U(x,0) = 0; \qquad U(0,t) = 1; \qquad U(1,t) = 0$$

Figure 2 shows the variation of U with respect to the space fractional parameter (α) and space coordinates (x) at kinematic viscosity $\mu = 0.1$ and 1.0 respectively. It is noted that, by increasing the parameter α, U decreases its intensity and travelling distance along the space.

4 Conclusion

The NS of SFBE has been evaluated by using implicit and Lax-Friedrichs-implicit FDM respectively. It is noted that both the implicit scheme is unconditionally stable and are good in agreement with the ES. It is found that L_2 of the Lax-Friedrichs-implicit is lesser than the implicit FDM. Also, it is found that the variation in space fractional order strongly affects the flow characteristics.

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Optimal Control of two-strain typhoid transmission using treatment and proper hygiene/sanitation practices

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Abstract

A mathematical model is developed to predict the optimum level of measures required to control a two-strain typhoid infection. The model considers symptomatic individuals and carriers together with environmental bacteria with different sensitivities to antimicrobials. Treatment for symptomatic individuals in each strain and use of sanitation and proper hygiene practices are considered as control measures. Our simulation results show that combining the three control interventions highly influenced the number of symptomatic individuals and environmental bacteria in both the strains. However, there are still a significant number of asymptomatic carriers in both the strains. This result shows that combating a two-strain typhoid infection requires some control interventions that reduce the number of asymptomatic carriers to near zero, along with optimal treatment combined with proper hygiene/sanitation practices. Further, efficiency analysis is used to investigate the impact of each control strategy on reducing the number of infected individuals and bacteria in both the strains. The study result suggests that implementing the combination of all the three control interventions is the most effective control strategy.

Key words: *Salmonella* Typhi; Two-strain typhoid infection; Asymptomatic carriers; Efficiency analysis

Mathematics Subject Classification(2010): 44A15; 46F12; 54B15; 46F99

1

1 Introduction

Typhoid, a disease caused by *Salmonella* Typhi bacteria, is a significant cause of illness and death in low-resource regions worldwide, especially Sub-Saharan

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Africa and South/Southeast Asia [1]. It is a severe febrile illness often accompanied by headache, loss of appetite, malaise, abdominal pain, diarrhea, and (in severe cases) intestinal perforation and neurological complications [2]. It is estimated to cause nearly 12 million cases and over 128 000 deaths globally each year [6]. It is estimated that the case fatality rate for untreated patients ranges between 10 and 20%, but drops to 1–4% with appropriate and timely antimicrobial treatment [4, 3, 5]. The infection is usually spread through contaminated food and water from the environment and direct contact with an infected person [7, 8].

Typhoid fever can be prevented and controlled through public health interventions such as providing safe drinking water, promoting hygiene and sanitation, and ensuring adequate and timely patient care. Antimicrobial treatment is the cornerstone for reducing severe illness and even death. However, misuse of antimicrobials for treatment leads to the emergence of resistant strains of *Salmonella* Typhi, known as treatment-induced acquired resistance [9, 10]. In typhoid endemic areas, clinicians frequently prescribe antimicrobials to patients with suspected typhoid without blood culture confirmation. This practice results in delayed treatment leading to the development of antimicrobial resistance [3, 11, 12, 13]. Treatment-induced acquired resistance has complicated treatment, increasing morbidity and mortality, and is considered one of the most significant challenges in managing the disease [14, 15].

In existing litrature, several typhoid epidemiological models have been developed and analyzed to better understand the transmission dynamics of typhoid [16, 17, 18, 19, 20, 21, 22, 23]. Among them, only a few have explored the effect of control strategies for typhoid with optimal control theory [16, 17]. Optimal control theory is a mathematical optimization that deals with finding a control for a dynamical system over a period of time. Although the importance of optimal control theory in epidemiology is well recognized, its applications in typhoid dynamics are scarce. No attempts have been made to predict the optimal level of control measures required to combat a two-strain typhoid infection. Our aim is to investigate the optimal control strategies in a two-strain dynamic model involving antimicrobial-sensitive and resistant strains of typhoid. A mathematical model for a two-strain typhoid dynamics is explored considering treatment-induced acquired resistance and re-infection [24]. Three time-dependent controls are introduced in this model to explore the optimal control strategy for controlling the disease.

The paper is organized as follows: In Section 2, the model in [24] is modified by adding three time-dependent controls $u_1(t)$, $u_2(t)$ and $u_3(t)$, and three positive parameters ϵ, b_1 and b_2. Also, a description of these parameters is given. In Section 3, a mathematical analysis of the time-dependent model is performed. In Section 4, numerical simulations and discussions of the corresponding results are presented. A short conclusion of the study is made in Section 5.

2 Model with controls

The mathematical model developed by Irena and Gakkhar [24] is considered to investigate the infection dynamics in a two-strain typhoid disease. The state variables I_j, C_j, and $\mathcal{B}_j$ represent the number of symptomatic infectious individuals, asymptomatic carriers, and bacteria for the strain j, respectively, while S represents the susceptible individuals. The model presented in [24] is

$$\begin{cases} \frac{dS}{dt} = \pi - \mu S - (\lambda_1 + \lambda_2)S + (1-p)r_1 I_1 + r_2 I_2 \\ \frac{dI_1}{dt} = (1-\alpha)[\lambda_1 S - \psi\lambda_2 I_1] - (\mu + d_1 + r_1)I_1 + \phi_1 C_1 \\ \frac{dC_1}{dt} = \alpha\lambda_1 S - \psi\lambda_2 C_1 - (\mu + \phi_1)C_1 \\ \frac{d\mathcal{B}_1}{dt} = \delta_1 I_1 + \omega_1 C_1 - \xi_1 \mathcal{B}_1 \\ \frac{dI_2}{dt} = (1-\alpha)\lambda_2[S + \psi(I_1 + C_1)] + pr_1 I_1 - (\mu + d_2 + r_2)I_2 + \phi_2 C_2 \\ \frac{dC_2}{dt} = \alpha\lambda_2(S + \psi C_1) - (\mu + \phi_2)C_2 \\ \frac{d\mathcal{B}_2}{dt} = \delta_2 I_2 + \omega_2 C_2 - \xi_2 \mathcal{B}_2 \end{cases} \tag{2.1}$$

where

$$\lambda_j = \frac{\beta_j(I_j + \theta C_j)}{N} + \eta f(B) g_j(B)$$

and $j = 1, 2$ represent the sensitive and resistant strains, respectively.

On the basis of sensitivity analysis of the model, three time-dependent controls are introduced in the model: (i) treatment of the symptomatic individuals in each strain $(u_1(t), u_2(t))$, which were constant parameters in our previous work [24], and (ii) proper hygiene/sanitation practices in order to prevent contamination of food and water to reduce both direct and environmental transmission $(u_3(t))$. The first two controls, u_1 and u_2, also decrease the bacteria excretion of symptomatic individuals in both strains so that the bacteria shedding rates by symptomatic individuals δ_1 and δ_2 in model (2.1) are replaced by $(1-(1-p)u_1)\delta_1$ and $(1-\epsilon u_2)\delta_2$, respectively. The parameter ϵ represents the efficacy of treatment for symptomatic individuals with resistant strain. Also, the second control u_3 increases the decay rate of bacteria so that the bacteria decay rates ξ_1 and ξ_1 are replaced by $\xi_1 + b_1 u_3$ and $\xi_2 + b_2 u_3$, respectively. The parameters b_1 and b_2 denote the bacteria decay rates (sensitive and AMR strains, respectively) induced by sanitation and proper hygiene practices. The schematic diagram in Figure 1 shows the transmission dynamics of the time-dependent model. Thus, the resulting dynamic model is given by the following

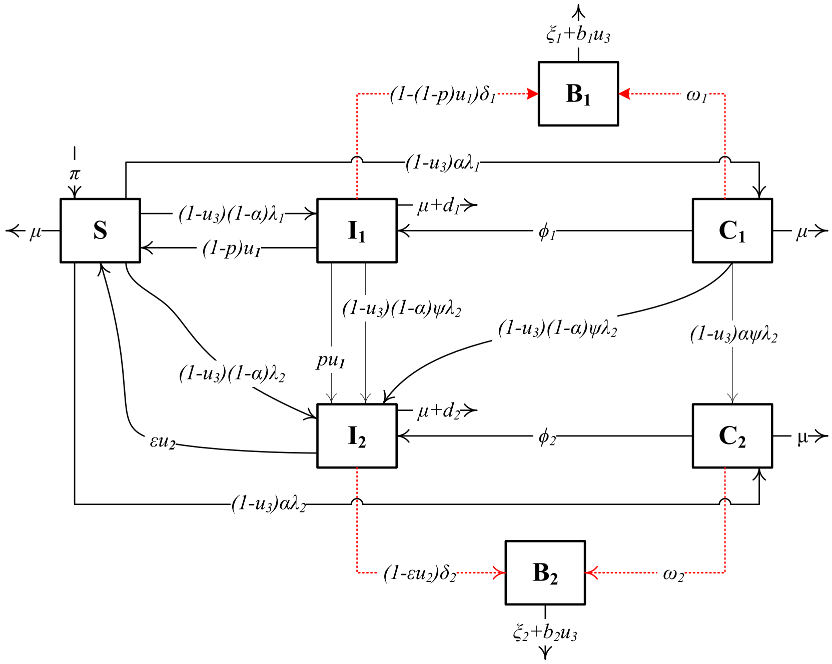

Figure 1: Flow diagram of the model.

system of nonlinear ODEs:

$$\begin{cases} \frac{dS}{dt} = \pi - \mu S - (1-u_3)(\lambda_1+\lambda_2)S + (1-p)u_1 I_1 + \epsilon u_2 I_2 \\ \frac{dI_1}{dt} = (1-u_3)(1-\alpha)[\lambda_1 S - \psi\lambda_2 I_1] - (\mu + d_1 + u_1)I_1 + \phi_1 C_1 \\ \frac{dC_1}{dt} = (1-u_3)[\alpha\lambda_1 S - \psi\lambda_2 C_1] - (\mu+\phi_1)C_1 \\ \frac{d\mathcal{B}_1}{dt} = \delta_1(1-(1-p)u_1)I_1 + \omega_1 C_1 - (\xi_1 + b_1 u_3)\mathcal{B}_1 \\ \frac{dI_2}{dt} = (1-\alpha)(1-u_3)\lambda_2[S + \psi(I_1 + C_1)] + pu_1 I_1 \\ \qquad -(\mu + d_2 + \epsilon u_2)I_2 + \phi_2 C_2 \\ \frac{dC_2}{dt} = (1-u_3)\alpha\lambda_2(S + \psi C_1) - (\mu+\phi_2)C_2 \\ \frac{d\mathcal{B}_2}{dt} = \delta_2(1-\epsilon u_2)I_2 + \omega_2 C_2 - (\xi_2 + b_2 u_3)\mathcal{B}_2 \end{cases} \tag{2.2}$$

The model is associated with the nonnegative initial conditions:

$$S(0), \quad I_j(0), \quad C_j(0), \quad \mathcal{B}_j(0) \ \text{ for } \ j = 1, 2.$$

The description of the associated model parameters are given in Table 1 and are assumed to be nonnegative.

Table 1: Description the model parameters.

Parameter	Description
α	Fraction of newly infected individuals who becomes asymptomatic carriers
β_1, β_2	Ingestion rate of sensitive and resistant strains of bacteria through human-to-human interaction
δ_1, δ_2	Shedding rate of bacteria by symptomatic cases with sensitive and resistant strains
ϵ	Efficacy of treatment of symptomatic individuals with resistant strain
η	Ingestion rate of bacteria from the contaminated environment
θ	Relative infectiousness of asymptomatic carriers
μ	Natural mortality rate of human population
ξ_1, ξ_2	Decay rate of sensitive and resistant strains of bacteria in the environment
π	Influx rate of individuals into susceptible class
ϕ_1, ϕ_2	Symptoms development rate by asymptomatic carriers with sensitive and resistant strains
ψ	Factor reducing the risk of re-infection with resistant strain due to activates of immune cells to the previous infection with a sensitive strain
ω_1, ω_2	Shedding rate of bacteria by asymptomatic carriers with sensitive and resistant strains
b_1, b_2	Sanitation-induced bacteria decay rates (sensitive and resistant strains)
d_1, d_2	Disease-induced death rate for symptomatic cases with sensitive and resistant strains
p	Fraction of those symptomatic individuals infected with a sensitive strain who acquire treatment-induced resistance

The objective functional to be minimized is

$$J(u_1, u_2, u_3) = \int_0^T \left(\sum_{i=1}^{2} A_i(I_i + C_i) + A_3 \sum_{j=1}^{2} \mathcal{B}_j + \frac{1}{2} \sum_{k=1}^{3} D_k u_k^2 \right) dt \tag{2.3}$$

subject to the state system (2.2), where A_i and D_i $(i = 1, 2, 3)$ are appropriate weight constants. The aim is to minimize the total number of infective individuals as well as bacteria while keeping the implementation cost of the strategies associated to the controls low.
We seek to find an optimal control triplet (u_1^*, u_2^*, u_3^*) such that

$$J(u_1^*, u_2^*, u_3^*) = \min_{\Omega} J(u_1, u_2, u_3)$$

where

$$\Omega = \left\{(u_1, u_2, u_3) \in L^1(0, T) \mid 0 \leq u_i \leq 1, \ i = 1, 2, 3\right\}$$

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is the control set.

3 Optimal control analysis

The existence of optimal control triplet (u_1^*, u_2^*, u_3^*) is guaranteed due to a priori boundedness of the state solutions, convexity of the integrand of J on Ω, and the *Lipschitz* property of the state system [25].

The necessary conditions that an optimal solution must satisfy come from Pontryagin's Maximum Principle [26]. This principle converts (2.2) and (2.3) into a problem of minimizing pointwise a Hamiltonian $\mathbb{H}$ with respect to u_1, u_2 and u_3:

$$\begin{aligned}
\mathbb{H} = {} & A_1(I_1 + C_1) + A_2(I_2 + C_2) + A_3(\mathcal{B}_1 + \mathcal{B}_2) + \frac{D_1}{2}u_1^2 + \frac{D_2}{2}u_2^2 + \frac{D_3}{2}u_3^2 \\
& +\lambda_1[\pi - \mu S - (1 - u_3)(\lambda_1 + \lambda_2)S + (1 - p)u_1 I_1 + \epsilon u_2 I_2] \\
& +\lambda_2[(1 - u_3)(1 - \alpha)(\lambda_1 S - \psi\lambda_2 I_1) - (\mu + d_1 + u_1)I_1 + \phi_1 C_1] \\
& +\lambda_3[(1 - u_3)(\alpha\lambda_1 S - \psi\lambda_2 C_1) - (\mu + \phi_1)C_1] \\
& +\lambda_4[\delta_1(1 - (1 - p)u_1)I_1 + \omega_1 C_1 - (\xi_1 + b_1 u_3)\mathcal{B}_1] \\
& +\lambda_5[(1 - u_3)(1 - \alpha)\lambda_2(S + \psi(I_1 + C_1)) + p u_1 I_1 \\
& \quad -(\mu + d_2 + \epsilon u_2)I_2 + \phi_2 C_2] \\
& +\lambda_6[(1 - u_3)\alpha\lambda_2(S + \psi C_1) - (\mu + \phi_2)C_2] \\
& +\lambda_7[\delta_2(1 - \epsilon u_2)I_2 + \omega_2 C_2 - (\xi_2 + b_2 u_3)\mathcal{B}_2]
\end{aligned} \tag{3.1}$$

where λ_i, $i = 1, 2, ..., 7$ are the adjoint functions.

By applying Pontryagin's Maximum Principle [26] and the existence result for the optimal control triplet from [25], the following adjoint system is obtained

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together with transversality conditions $\lambda_k(T) = 0$:

$$\begin{aligned}
\frac{d\lambda_1}{dt} =& \frac{1}{\mathcal{B}_1 + \mathcal{B}_2} \\
& \times [(1-u_3)\eta f(\mathcal{B})((\lambda_1 - (1-\alpha)\lambda_2 - \alpha\lambda_3)\mathcal{B}_1 + (\lambda_1 - (1-\alpha)\lambda_5 - \alpha\lambda_6)\mathcal{B}_2) \\
& + \mu(\mathcal{B}_1 + \mathcal{B}_2)\lambda_1] \\
& + \frac{(1-u_3)}{N^2}[\mathcal{B}_1(\lambda_1 - (1-\alpha)\lambda_2 - \alpha\lambda_3)(I_1 + \theta C_1)(I_1 + C_1 + I_2 + C_2)] \\
& + \frac{\beta_2(1-u_3)(I_2+\theta C_2)}{N^2}[(\lambda_1 - (1-\alpha)\lambda_5 - \alpha\lambda_6 + (\lambda_5 - \lambda_2)(1-\alpha)\psi)I_1 \\
& + (\lambda_1 - (\lambda_5(1-\alpha) + \alpha\lambda - 6)(1-\psi) - \psi\lambda_3)C_1 \\
& + (\lambda_1 - (1-\alpha)\lambda_5 - \alpha\lambda_6)I_2 + (\lambda_1 - (1-\alpha)\lambda_5 - \alpha\lambda_6)C_2], \\
\frac{d\lambda_2}{dt} =& -A_1 - [(1-p)u_1 - S(1-u_3)\frac{\beta_1(S + (1-\theta)C_1 + I_2 + C_2) - \beta_2(I_2 + \theta C_2)}{N^2}]\lambda_1 \\
& + (\mu + d_1 + u_1)\lambda_2 \\
& - (1-u_3)(1-\alpha)[\frac{\beta_1 S(S + (1-\theta)C_1 + I_2 + C_2)}{N^2} - \frac{\beta_2\psi(S + C_1 + I_2 + C_2)(I_2 + \theta C_2)}{N^2} \\
& - \frac{\psi\eta f(\mathcal{B})\mathcal{B}_2}{\mathcal{B}_1 + \mathcal{B}_2}]\lambda_2 - (1-u_3)[\frac{\alpha\beta_1 S(S + (1-\theta)C_1 + I_2 + C_2)}{N^2} + \frac{\beta_2\psi C_1(I_2 + \theta C_2)}{N^2}]\lambda_3 \\
& - (1-u_1)\delta_1\lambda_4 - pu_1\lambda_5 \\
& - (1-u_3)(1-\alpha)[\frac{\beta_2(I_2+\theta C_2)(S(\psi-1) + \psi(I_2 + C_2))}{N^2} + \frac{\psi\eta f(\mathcal{B})\mathcal{B}_2}{\mathcal{B}_1 + \mathcal{B}_2}]\lambda_5 \\
& + \frac{(1-u_3)\alpha\beta_2(S + \psi C_1)(I_2 + \theta C_2)}{N^2}\lambda_6, \\
\frac{d\lambda_3}{dt} =& -A_1 + S(1-u_3)\frac{\beta_1(-I_1 + \theta(S + I_1 + I_2 + C_2)) - \beta_2(I_2 + \theta C_2)}{N^2}\lambda_1 + (\mu + \phi_1)\lambda_3 \\
& + [(1-u_3)(1-\alpha)\frac{\beta_1 S(-I_1 + \theta(S + I_1 + I_2 + C_2)) + \beta_2\psi I_1(I_2 + \theta C_2)}{N^2} + \phi_1]\lambda_2 \\
& - (1-u_3)\frac{\alpha\beta_1 S(-I_1 + \theta(S + I_1 + I_2 + C_2)) - \beta_2\psi(I_2 + \theta C_2)(S + I_1 + I_2 + C_2)}{N^2}\lambda_3 \\
& + (1-u_3)\frac{\psi\eta f(\mathcal{B})\mathcal{B}_2}{\mathcal{B}_1 + \mathcal{B}_2}\lambda_3 - \omega_1\lambda_4 \\
& - (1-u_3)(1-\alpha)[\frac{\beta_2(I_2+\theta C_2)(S(\psi-1) + \psi(I_2 + C_2))}{N^2} + \frac{\psi\eta f(\mathcal{B})\mathcal{B}_2}{\mathcal{B}_1 + \mathcal{B}_2}]\lambda_5 \\
& - (1-u_3)\alpha[\frac{\beta_2(I_2+\theta C_2)(S(\psi-1) + \psi(I_1 + I_2 + C_2))}{N^2} + \frac{\psi\eta f(\mathcal{B})\mathcal{B}_2}{\mathcal{B}_1 + \mathcal{B}_2}]\lambda_6,
\end{aligned}$$

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$$
\begin{aligned}
\frac{d\lambda_4}{dt} =& - A_3 + \frac{(1-u_3)\eta S}{(1+\mathcal{B}_1)^2(1+\mathcal{B}_2)}\lambda_1 + (\xi_1 + b_1 u_3)\lambda_4 \\
& - \frac{(1-u_3)\eta f(\mathcal{B})\mathcal{B}_2}{(\mathcal{B}_1+\mathcal{B}_2)^2}[(1-\alpha)((S+\psi I_1)\lambda_2 - (S+\psi(I_1+C_1))\lambda_5) \\
& + (\alpha S + \psi C_1)\lambda_3 - \alpha(S+\psi C_1)\lambda_6] - (1-u_3)\eta[\frac{\mathcal{B}_1 S((1-\alpha)\lambda_2 + \alpha\lambda_3)}{(1+\mathcal{B}_1)^2(1+\mathcal{B}_2)(\mathcal{B}_1+\mathcal{B}_2)} \\
& + \frac{\mathcal{B}_2\{-\psi C_1\lambda_3 + \alpha\lambda_6(S+\psi C_1) + \psi(-1+\alpha)\lambda_2 I_1 + (1-\alpha)\lambda_5\ (S+\psi(C_1+I_1))\}}{(1+\mathcal{B}_1)^2(1+\mathcal{B}_2)(\mathcal{B}_1+\mathcal{B}_2)}], \\
\frac{d\lambda_5}{dt} =& - A_2 - [u_2 + S(1-u_3)\frac{\beta_1(I_1+\theta C_1) - \beta_2(S+I_1+C_1+(1-\theta)C_2)}{N^2}]\lambda_1 \\
& + (1-u3)(1-\alpha)\frac{\beta_1 S(I_1+\theta C_1) + \beta_2\psi I_1(S+I_1+C_1+(1-\theta)C_2)}{N^2}\lambda_2 \\
& + (1-u3)\frac{\beta_1\alpha S(I_1+\theta C_1) + \beta_2\psi C_1(S+I_1+C_1+(1-\theta)C_2)}{N^2}\lambda_3 \\
& + [(\mu+d_2+u_2) - \frac{(1-u_3)(1-\alpha)(S+I_1+C_1+(1-\theta)C_2)(S+\psi(I_1+C_1))\beta_2}{N^2}]\lambda_5 \\
& - \frac{(1-u_3)\alpha(S+\psi C_1)(S+I-1+C_1+(1-\theta)C_2)\beta_2}{N^2}\lambda_6 - (1-\epsilon u_2)\delta_2\lambda_7, \\
\frac{d\lambda_6}{dt} =& - A_2 + S(1-u_3)\frac{\beta_2(-I_2+\theta(S+I_1+C_1+I_2)) - \beta_1(I_1+\theta C_1)}{N^2}\lambda_1 \\
& + (1-u_3)(1-\alpha)\frac{\beta_1 S(I_1+\theta C_1) + \beta_2\psi I_1(-I_2+\theta(C_1+S+I_1+I_2))}{N^2}\lambda_2 \\
& + (1-u_3)\frac{\alpha\beta_1 S(I_1+\theta C_1) + \beta_2\psi C_1(-I_2+\theta(S+I_1+C_1+I_2))}{N^2}\lambda_3 \\
& - [\frac{(1-u_3)(1-\alpha)(S+\psi(I_1+C_1))(-I_2+\theta(S+I_1+C_1+I_2))\beta_2}{N^2} + \phi_2]\lambda_5 + (\mu+\phi_2)\lambda_6 \\
& - \frac{(1-u_3)\alpha(S+\psi C_1)(-I_2+\theta(S+I_1+C_1+I_2))\beta_2}{N^2}\lambda_6 - \omega_2\lambda_7, \\
\frac{d\lambda_7}{dt} =& - A_3 + \frac{(1-u_3)\eta S}{(1+\mathcal{B}_1)(1+\mathcal{B}_2)^2}\lambda_1 + (\xi_1 + b_1 u_3)\lambda_4 \\
& - \frac{(1-u_3)\eta f(\mathcal{B})\mathcal{B}_1}{(\mathcal{B}_1+\mathcal{B}_2)^2}[(1-\alpha)((S+\psi I_1)\lambda_2 - (S+\psi(I_1+C_1))\lambda_5) \\
& + (\alpha S + \psi C_1)\lambda_3 - \alpha(S+\psi C_1)\lambda_6] - (1-u_3)\eta[\frac{\mathcal{B}_1 S((1-\alpha)\lambda_2 + \alpha\lambda_3)}{(1+\mathcal{B}_1)(1+\mathcal{B}_2)^2(\mathcal{B}_1+\mathcal{B}_2)} \\
& + \frac{\mathcal{B}_2\{-\psi C_1\lambda_3 + \alpha\lambda_6(S+\psi C_1) + \psi(-1+\alpha)\lambda_2 I_1 + (1-\alpha)\lambda_5(S+\psi(C_1+I_1))\}}{(1+\mathcal{B}_1)(1+\mathcal{B}_2)^2(\mathcal{B}_1+\mathcal{B}_2)}]
\end{aligned}
\tag{3.2}
$$

Furthermore, the optimal control characterization is

$$\begin{aligned}
u_1^* &= max\left\{0, min\left(\frac{(\lambda_2 + (1-p)(\delta_1\lambda_4 - \lambda_1) - p\lambda_5)I_1^*}{D_1}, 1\right)\right\} \\
u_2^* &= max\left\{0, min\left(\frac{\epsilon(\lambda_5 + \delta_2\lambda_7 - \lambda_1)I_2^*}{D_2}, 1\right)\right\} \\
u_3^* &= max\left\{0, min\left(\tilde{u}_3, 1\right)\right\}
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
\tilde{u}_3 = \frac{\eta f(\mathcal{B})}{D_3}&[-\lambda_1 S^* + \frac{S^*\mathcal{B}_1^*(1-\alpha)\lambda_2 + \alpha\lambda_3}{\mathcal{B}_1^* + \mathcal{B}_2^*} \\
&+ \frac{\mathcal{B}_2^*\left(\psi(1-\alpha)(\lambda_5 - \lambda_2)I_1^* + ((1-\alpha)\lambda_5 + \alpha\lambda_6)(S^* + \psi C_1^*) - \psi\lambda_3 C_1^*\right)}{\mathcal{B}_1^* + \mathcal{B}_2^*}] \\
&+ \frac{b_1\lambda_4\mathcal{B}_1^* + b_2\lambda_7\mathcal{B}_2^*}{D_3} - \frac{\beta_1}{D_3 N^*}(I_1^* + \theta C_1^*)(\lambda_1 - \lambda_2 + \alpha(\lambda_2 - \lambda_3))S^* \\
&- \frac{\beta_2}{D_3 N^*}(I_2^* + \theta C_2^*)[\lambda_1 S^* + \psi\lambda_3 C_1^* + (S^* + \psi C_1^*)(-\lambda_5 + \alpha(\lambda_5 - \lambda_6)) \\
&+ \psi(1-\alpha)(\lambda_2 - \lambda_5)I_1^*].
\end{aligned}$$

4 Numerical results

This section presents the numerical simulation results by solving the optimality system, which comprises the state system (2.2), adjoint system (3.2), control characterization (3.3), and corresponding initial and final conditions, using the forward-backward sweep method [27, 28].

For numerical simulations, we consider the model parameter values presented in Table 2.

Table 2: Model parameter values used in numerical simulations [24], the unit is per week if appropriate.

$\alpha = 0.3$	$\beta_1 = 0.006$	$\beta_2 = 0.0052$	$\delta_1 = 1.0$
$\delta_2 = 1.05$	$\eta = 1.379 \times 10^{-10}$	$\theta = 0.35$	$\mu = 0.0005$
$\xi_1 = 0.2415$	$\xi_2 = 0.2415$	$\pi = 10^5/52$	$\phi_1 = 0.00096$
$\phi_2 = 0.0017$	$\psi = 0.95$	$\omega_1 = 0.05$	$\omega_2 = 0.06$
$d_1 = 0.00125$	$d_2 = 0.002$	$p = 0.1$	

Additionally, the following parameter values are chosen:

$$A_1 = A_2 = 10,\ A_3 = 25,\ D_1 = 5,\ D_2 = 8,\ D_3 = 10,\ b_1 = 0.2,\ b_2 = 0.1,$$
$$\epsilon = 0.75,\ T = 100 \text{ weeks}.$$

The following control strategies are explored in order to determine the optimum strategy that significantly reduces typhoid transmission:

A: Treatment of the symptomatic individuals in each strain (u_1, u_2) only;

B: Employing sanitation and proper hygiene (u_3) only;

C: Employing all the three control interventions $(u_1,\ u_2,\ u_3)$.

The control profile for each control strategy is shown in Fig. 2, and the effect of each control strategy on the reduction of infection is depicted in Fig. 3.

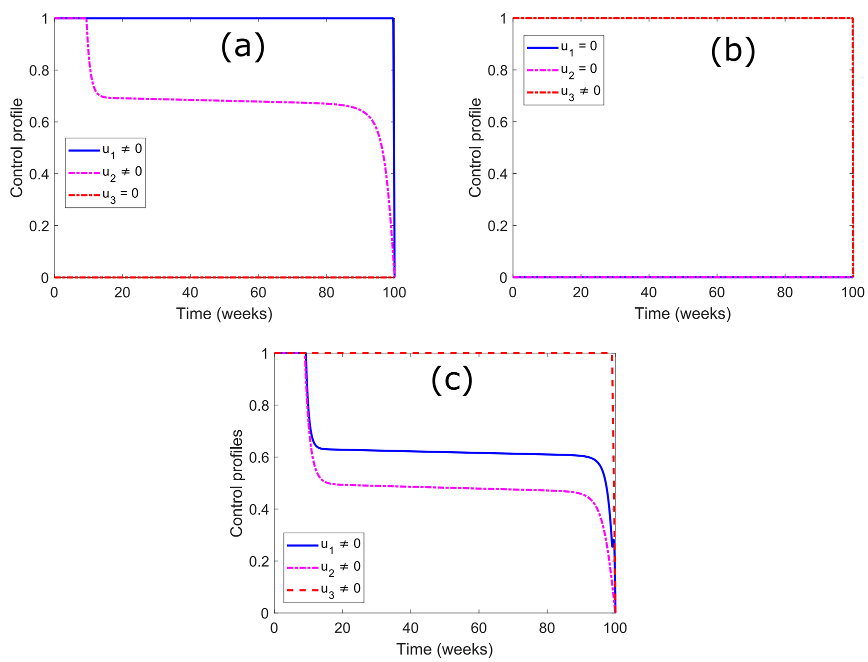

Figure 2: Control profile for (a) optimal treatment only, (b) optimal sanitation and proper hygiene only, and (c) optimal treatment combined with sanitation and proper hygiene

Our simulation results reveal that the combination of all control interventions highly influenced the symptomatic individuals and environmental bacteria in both the strains. However, there are still a significant number of asymptomatic carriers in both the strains, which play an important role in the evolution and transmission of typhoid infections. This reflects that asymptomatic carriers may have long-term impacts on the spread of typhoid infection even in the presence of the two control interventions.

4.1 Efficiency analysis

Here an efficiency analysis is performed to determine the best control strategy without considering costs associated with each control strategy [29, 30]. So, we

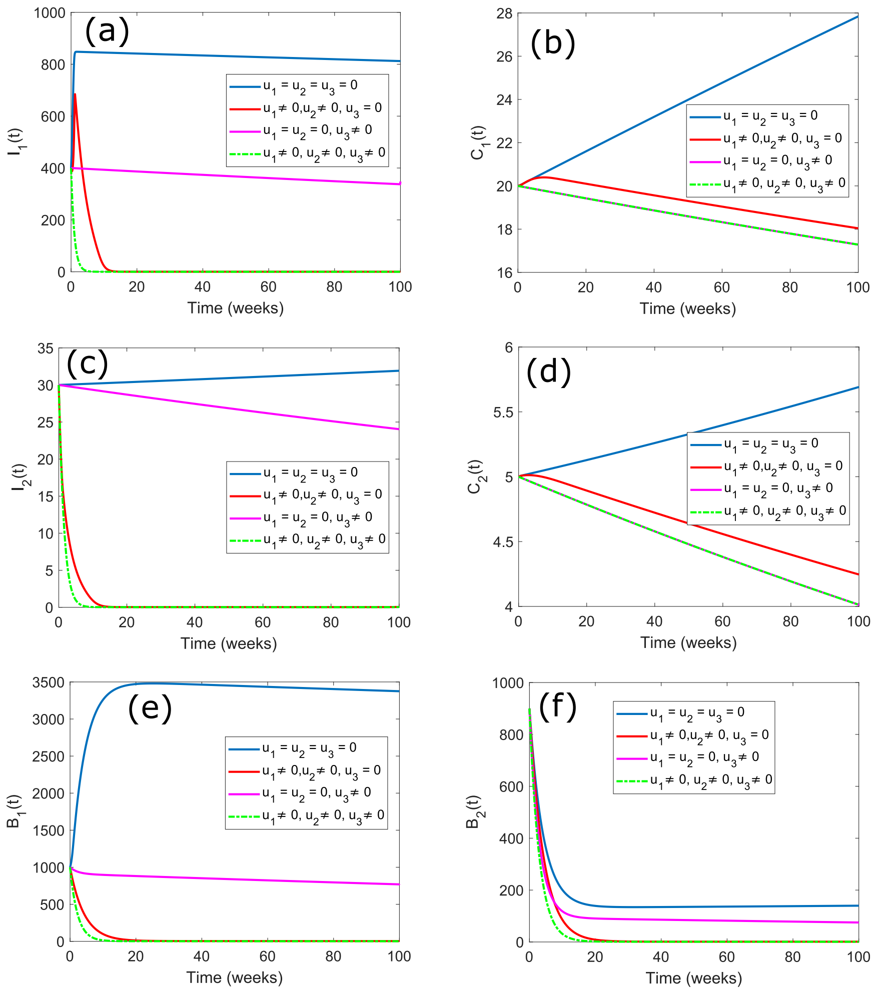

Figure 3: Effect of each control strategy on reducing the number of infectious humans and bacteria: (a) Symptomatic individuals with sensitive strain, (b) Asymptomatic carriers with sensitive strain, (c) Symptomatic individuals with resistant strain, (d) Asymptomatic carriers with resistant strain, (e) Sensitive strain of bacteria in the environment, (f) Resistant strain of bacteria in the environment

investigate the impact of each control strategies on the reduction of infectious humans and bacteria by introducing the efficiency index, $\mathbb{F}$. The efficiency index for human and bacteria population in the strain j are, respectively, computed

as:

$$\mathbb{F}^{I_j+C_j} = \left(1 - \frac{A_c^{I_j+C_j}}{A_o^{I_j+C_j}}\right) \times 100 \quad \text{and} \quad \mathbb{F}^{\mathcal{B}_j} = \left(1 - \frac{A_c^{\mathcal{B}_j}}{A_o^{\mathcal{B}_j}}\right) \times 100$$

where

$$A^{I_j+C_j} = \int_0^T \Big(I_j(t) + C_j(t)\Big)dt \quad \text{and} \quad A^{\mathcal{B}_j} = \int_0^T \mathcal{B}_j(t)dt$$

represent the cumulative number of infectious humans and bacteria with strain j, respectively, during the time interval $[0, T]$. The efficiency index is calculated for human and bacteria population in both the strains and presented in Table 3. Note that the control strategy with the highest efficiency index will be the best. From Table 3, it follows that strategy C is the most effective for reducing

Table 3: Efficiency index

Strategy	$A_c^{I_1+C_1}$	$A_c^{\mathcal{B}_1}$	$\mathbb{F}^{I_1+C_1}$	$\mathbb{F}^{\mathcal{B}_1}$	$A_c^{I_2+C_2}$	$A_c^{\mathcal{B}_2}$	$\mathbb{F}^{I_2+C_2}$	$\mathbb{F}^{\mathcal{B}_2}$
No control	87084	340679	0.0	0.0	3617	16767	0.0	0.0
A	1958	4567	97.75	98.66	508	3933	85.96	76.54
B	6541	12887	92.45	96.22	3143	10817	13.11	35.49
C	1918	2525	97.80	99.23	493	2792	86.37	83.35

the disease burden, followed by strategy A and strategy B.

5 Conclusions

The novelty of this study is its ability to predict the optimal level of control interventions that include treatment and proper hygiene/ sanitation practices. On the basis of sensitivity analysis of a two-strain typhoid model incorporating symptomatic infection, asymptomatic carriers, and environmental bacteria, some control measures were suggested in [24]. Accordingly, the time-dependent functions representing the treatment of sensitive and resistant strains are considered as control measures. Proper hygiene and sanitation are also considered as another control measure to prevent contamination of food and water. The necessary and sufficient conditions for the existence of optimal controls are established and the optimality system is developed. The characterization of the optimal control is determined by the Pontryagin's maximum principle. The numerical simulations are performed for every single control and combination of the two controls. The simulation results reveal that with the combination of the two control interventions, the number of symptomatic individuals and doses of *S.* Typhi bacteria in both the strains reduced to near zero. However, there is still a significant number of asymptomatic carriers in both strains, which play an essential role in the evolution and transmission of typhoid infections. So, additional preventive measures need to be implemented in order to further reduce the population of asymptomatic carriers. The effects of each control strategy on

the reduction of infection in both the strains is investigated through efficiency analysis. From the study results, we conclude that the fight against a two-strain typhoid infection requires some control interventions that reduce the number of asymptomatic carriers to near zero, along with optimal treatment combined with sanitation and proper hygiene.

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Bicomplex Laplace Transform of Fractional Order, Properties and applications

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Abstract

The aim of this research article is to define bicomplex Laplace transform of fractional order or fractional Laplace transform by the application of the Mittag-Leffler function. Various properties of bicomplex fractional Laplace transform along with the convolution theorem have also been given. Inverse bicomplex fractional Laplace transform has also been defined. Application of bicomplex fractional Laplace transform in the solution of diffusion equation has been given.

Key words: Bicomplex numbers, Fractional derivative, fractional Laplace transform, Mittag-Leffler function.
Mathematics Subject Classification(2010): 30G35, 44A10, 33E12.

1 Introduction

In recent years, mathematicians and physicists have focused their efforts on bicomplex algebra. In 1882, Segre [25] introduced bicomplex numbers. Detailed study of bicomplex numbers are presented by Riley [20], Price [18], Rönn [24]. A bicomplex number is defined as an ordered pair of complex numbers, similar like how a complex number is defined as an ordered pair of real numbers.

In recent years, the fractional order differential equations with boundary conditions have gained more attention in a variety of scientific and engineering domains. The Mittag-Leffler function (see, e.g. [7, 10]) has an important contribution in the study of fractional calculus, it has been used to solve fractional order differential equations. The Mittag-Leffler function has caught the interest of a number of authors working in the field of fractional calculus (FC) and its applications such as, usage of a fractional operator involving Mittag–Leffler function for the generalized Casson fluid flow [29], to established the fractional calculus operators with Appell function kernels and Caputo-type fractional differential operators [16], Epidemiological analysis of fractional order COVID-19 model with Mittag-Leffler kernel [6]. In recent developments authors have worked on

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the area of fractional calculus such as, to study a guava fruit model associated with a non-local additionally non-singular fractional derivative [27], the approximate solution of nonlinear Caudrey-Dodd-Gibbon equation of fractional order [28], analysis of fractional blood alcohol model [26].

Many authors have studied the applications of the fractional integral transform [9, 13, 14, 17, 23]. Efforts have been made by authors to introduce the Mittag-Leffler function (ML function) in bicomplex space along with applications to fractional calculus and integral transform [4, 5]. In 2011 bicomplex Laplace transform is introduced by Kumar et al. [15] and its convolution theorem and applications in bicomplex space are discussed by Agarwal et al. [1], bicomplex double Laplace transform is derived by Goswami et al. [8].

Following the path, efforts are made to extend the fractional Laplace transform in bicomplex space. Fractional Laplace transformation method is a effective and strong tool for finding a solution of the fractional differential equation. In this article bicomplex fractional Laplace transform and its properties in bicomplex space are introduced.

2 Preliminaries

2.1 Bicomplex Numbers

Definition 2.1 (Bicomplex Number)**.** A bicomplex number $\xi \in \mathbb{T}$ can be written as [25]

$$\xi = x_0 + i_1 x_1 + i_2 x_2 + j x_3, \text{ where } x_0,\ x_1,\ x_2,\ x_3\ \in \mathbb{R}. \tag{2.1}$$

Here $\mathbb{T}$, $\mathbb{R}$ represents the set of bicomplex numbers and real numbers respectively.

We shall use the notations, $x_0 = \mathrm{Re}(\xi),\ x_1 = \mathrm{Im}_{i_1}(\xi),\ x_2 = \mathrm{Im}_{i_2}(\xi),\ x_3 = \mathrm{Im}_j(\xi)$.

Idempotent representation is particularly important since it allows for term-by-term addition, multiplication, and division.

Definition 2.2 (Idempotent Representation)**.** Every bicomplex number has following idempotent representation [18]

$$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2. \tag{2.2}$$

Hence if $\xi_1 = (z_1 - i_1 z_2)$ and $\xi_2 = (z_1 + i_1 z_2)$ then

$$\xi = \xi_1 e_1 + \xi_2 e_2, \tag{2.3}$$

where $e_1,\ e_2$ are idempotent elements in $\mathbb{T}$ such that $e_1 = \dfrac{1 + i_1 i_2}{2} = \dfrac{1+j}{2}$, $e_2 = \dfrac{1 - i_1 i_2}{2} = \dfrac{1-j}{2}$ and $e_1 + e_2 = 1,\ e_1.e_2 = 0$.

Projection Mappings

$P_1 : \mathbb{T} \to T_1 \subseteq \mathbb{C}$, $P_2 : \mathbb{T} \to T_2 \subseteq \mathbb{C}$ for a bicomplex number $\xi = z_1 + i_2 z_2$ are

given by (see, e.g. [2, 22]):

$$P_1(\xi) = P_1(z_1 + i_2 z_2) = (z_1 - i_1 z_2) \in T_1, \tag{2.4}$$

and

$$P_2(\xi) = P_2(z_1 + i_2 z_2)(z_1 + i_1 z_2) \in T_2, \tag{2.5}$$

where

$$T_1 = \{\xi_1 = z_1 - i_1 z_2 \,|z_1, z_2 \in \mathbb{C}\} \text{ and } T_2 = \{\xi_2 = z_1 + i_1 z_2 \,|z_1, z_2 \in \mathbb{C}\}. \tag{2.6}$$

2.2 Bicomplex One-Parameter Mittag-Leffler Function

The bicomplex one parameter ML function defined Agarwal et al. [5] is given by

$$\mathbb{E}_\alpha(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{\Gamma(\alpha n + 1)}, \tag{2.7}$$

where $\xi, \alpha \in \mathbb{T},\ \xi = z_1 + i_2 z_2$ and $|\mathrm{Im_j}(\alpha)| < \mathrm{Re}(\alpha)$.

2.3 Modified Riemann- Liouville Derivative

Definition 2.3 (Modified Riemann-Liouville Derivative, [12]). Let $g : \mathbb{R} \to \mathbb{R},\ y \to g(y)$ represents a continuous function (not necessarily differentiable) function

1. If $g(y)$ is a constant M then its fractional derivative of order μ is given by

$$^J D_y^\mu M = \begin{cases} \frac{M}{\Gamma(1-\mu) y^\mu} & if\ \mu \le 0, \\ 0 & if\ \mu > 0. \end{cases}$$

2. If $g(y)$ is not a constant then its fractional derivative of order μ is given by

$$^J D_y^\mu (g(y) - g(0)) = \frac{1}{\Gamma(-\mu)} \int_0^y \frac{g(\zeta) d\zeta}{(y-\zeta)^{\mu+1}},\ \mu < 0, \tag{2.8}$$

$$^J D_y^\mu (g(y) - g(0)) = {}^J D_y^\mu g(y) = {}^J D_y (g^{\mu-1}(y)),\ \mu > 0, \tag{2.9}$$

$$(g^\mu(y)) = \left(g^{\mu-n}(y)\right)^{(n)},\ n \le \mu \le n+1,\ n \ge 1. \tag{2.10}$$

2.4 Laplace Transform of Fractional Order

Let $g(x)$ denotes the function which vanishes for negative values of the variable x. Its Laplace transform (LT) of order α is defined by the expression (see, e.g. [13, 14, 19]), when it is finite,

$$L_\alpha(g(x)) = \int_0^\infty E_\alpha(-s^\alpha x^\alpha) g(x) (dx)^\alpha,\ 0 < \alpha < 1, \tag{2.11}$$

where $s \in \mathbb{C}$.
Sufficient condition for this integral to be finite is that (see, e.g.[13])

$$\int_0^\infty |g(x)|(dx)^\alpha < M < \infty. \tag{2.12}$$

If $g(u)$ is a continuous function, the integral with respect to $(du)^\alpha$ is defined as (see, e.g. [14]) the fractional differential equation's solution $y(u)$

$$dy = g(t)(du)^\alpha,\ x \geq 0,\ y(0) = 0, \tag{2.13}$$

where

$$y = \int_0^u g(v)(dv)^\alpha = \alpha \int_0^u \frac{g(v)}{(u-v)^{1-\alpha}} dv,\ 0 < \alpha < 1. \tag{2.14}$$

Jumarie [11] gave the the proof of the above result as follows:

$$x^{(\alpha)}(u) = g(u),\ 0 < \alpha \leq 1. \tag{2.15}$$

Its solution is obtained by fractional derivative as

$$x(u) = D^{-\alpha} g(u) = \frac{1}{\Gamma\alpha} \int_0^u (u-t)^{\alpha-1} g(t) dt. \tag{2.16}$$

Again

$$d^\alpha x = g(u)(du)^\alpha, \tag{2.17}$$

or

$$\Gamma(\alpha+1) dx = g(u)(du)^\alpha. \tag{2.18}$$

On integrating

$$x(u) = \frac{1}{\Gamma(\alpha+1)} \int_0^u g(t)(dt)^\alpha. \tag{2.19}$$

From equations (2.16) and (2.19), equation (2.14) can be obtained.

3 Bicomplex Laplace transform of Fractional order

In this section we introduce the bicomplex fractional Laplace transform with convergence conditions using the bicomplex ML function.

Definition 3.1 (Class $\mathcal{C}$)**.** Let $\mathcal{C}$ be the class of bicomplex-valued functions defined with the following properties, for any $f \in \mathcal{C}$

1. $f(x)$ vanishes for negative values of the variable x.
2. f is piecewise continuous in the interval $(0, a]$ for any $a \in (0, +\infty)$.

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3. $\int_0^\infty |f(x)|_j (dx)^\alpha < M < \infty$.

Now we introduce the bicomplex Laplace transform of fractional order α as follows:
Let Laplace transform of order α of $f(t) \in \mathcal{C}$ for $t \geq 0$ can be written as

$$L_\alpha(f(t))_{s_1} = F_\alpha(s_1) = \int_0^\infty E_\alpha(-s_1^\alpha t^\alpha) f(t)(dt)^\alpha,\ 0 < \alpha < 1, \tag{3.1}$$

where $s_1 \in \mathbb{C}$ and take another LT of order α of $f(t) \in \mathcal{C}$ for $s_2 \in \mathbb{C}$

$$L_\alpha(f(t))_{s_2} = F_\alpha(s_2) = \int_0^\infty E_\alpha(-s_2^\alpha t^\alpha) f(t)(dt)^\alpha,\ 0 < \alpha < 1. \tag{3.2}$$

Now we take linear combination of $F_\alpha(s_1)$ and $F_\alpha(s_2)$ with e_1 and e_2 such as

$$\begin{aligned}
&L_\alpha(f(t))_{s_1} e_1 + L_\alpha(f(t))_{s_2} e_2 \\
&\quad = F_\alpha(s_1) e_1 + F_\alpha(s_2) e_2 \\
&\quad = \int_0^\infty E_\alpha(-s_1^\alpha t^\alpha) f(t)(dt)^\alpha e_1 + \int_0^\infty E_\alpha(-s_2^\alpha t^\alpha) f(t)(dt)^\alpha e_2 \\
&\quad = \int_0^\infty \left(E_\alpha(-s_1^\alpha t^\alpha) e_1 + E_\alpha(-s_2^\alpha t^\alpha) e_2\right) f(t)(dt)^\alpha \\
&\quad = \int_0^\infty E_\alpha(-\xi^\alpha t^\alpha) f(t)(dt)^\alpha \\
&\quad = F_\alpha(\xi) \\
&\quad = L_\alpha(f(t))_\xi,
\end{aligned} \tag{3.3}$$

where $\xi = s_1 e_1 + s_2 e_2 \in \mathbb{T}$.
Since $F_\alpha(s_1)$ and $F_\alpha(s_2)$ are complex valued functions which are convergent and analytic for respectively, so by application of decomposition theorem of Ringleb [21], (see, e.g. [20]) bicomplex valued function $F_\alpha(\xi) = F_\alpha(s_1)e_1 + F_\alpha(s_2)e_2$ will be convergent and analytic.

Definition 3.2 (Bicomplex Laplace Transform of Fractional Order)**.** Let $g(t) \in \mathcal{C}$ be a bicomplex valued function. Then bicomplex Laplace transform of fractional order α of $g(t)$ for $t \geq 0$ can be defined as

$$L_\alpha(g(t))_\xi = G_\alpha(\xi) = \int_0^\infty E_\alpha(-\xi^\alpha t^\alpha) g(t)(dt)^\alpha = \lim_{\mathcal{M}\to\infty} \int_0^{\mathcal{M}} E_\alpha(-\xi^\alpha t^\alpha) g(t)(dt)^\alpha, \tag{3.4}$$

where $0 < \alpha < 1,\ \xi = s_1 e_1 + s_2 e_2 \in \mathbb{T},\ s_1,\ s_2 \in \mathbb{C}$.

3.1 Some Basic Properties of Bicomplex Fractional Laplace Transform

Theorem 3.3 (Linearity Property)**.** *Let $F_\alpha(\xi)$ and $G_\alpha(\xi)$ be the bicomplex fractional Laplace transform of order α of class $\mathcal{C}$ functions $f(t)$ and $g(t)$ respectively, then*

$$L_\alpha\left(f(t)+g(t)\right)=F_\alpha(\xi)+G_\alpha(\xi). \tag{3.5}$$

Proof. Let $\xi=s_1e_1+s_2e_2\in\mathbb{T},\ s_1,\ s_2\in\mathbb{C}$ and $0<\alpha<1$ then

$$\begin{aligned}
L_\alpha\left(f(t)+g(t)\right) &= \int_0^\infty E_\alpha(-\xi^\alpha t^\alpha)\left(f(t)+g(t)\right)(dt)^\alpha \\
&= \int_0^\infty E_\alpha(-s_1^\alpha t^\alpha)\left(f_1(t)+g_1(t)\right)(dt)^\alpha e_1 \\
&\quad + \int_0^\infty E_\alpha(-s_2^\alpha t^\alpha)\left(f_2(t)+g_2(t)\right)(dt)^\alpha e_2 \\
&= \int_0^\infty E_\alpha(-s_1^\alpha t^\alpha)f_1(t)(dt)^\alpha e_1+\int_0^\infty E_\alpha(-s_1^\alpha t^\alpha)g_1(t)(dt)^\alpha e_1 \\
&\quad + \int_0^\infty E_\alpha(-s_2^\alpha t^\alpha)f_2(t)(dt)^\alpha e_2+\int_0^\infty E_\alpha(-s_2^\alpha t^\alpha)g_2(t)(dt)^\alpha e_2 \\
&= \int_0^\infty E_\alpha(-\xi^\alpha t^\alpha)f(t)(dt)^\alpha+\int_0^\infty E_\alpha(-\xi^\alpha t^\alpha)g(t)(dt)^\alpha \\
&= L_\alpha\left(f(t)\right)+L_\alpha\left(g(t)\right) \\
&= F_\alpha(\xi)+G_\alpha(\xi).
\end{aligned} \tag{3.6}$$

□

Theorem 3.4. *Let $F_\alpha(\xi)$ be the bicomplex fractional Laplace transform of order α of function $f(t)\in\mathcal{C}$ and $\xi=s_1e_1+s_2e_2\in\mathbb{T},\ s_1,\ s_2\in\mathbb{C},\ 0<\alpha<1$ and k is a constant then*

$$L_\alpha\left(kf(t)\right)=kF_\alpha(\xi). \tag{3.7}$$

Proof. Let $\xi=s_1e_1+s_2e_2\in\mathbb{T},\ s_1,\ s_2\in\mathbb{C}$ and $0<\alpha<1$, then

$$\begin{aligned}
L_\alpha\left(kf(t)\right) &= \int_0^\infty E_\alpha(-\xi^\alpha t^\alpha)\left(kf(t)\right)(dt)^\alpha \\
&= \int_0^\infty E_\alpha(-s_1^\alpha t^\alpha)\left(kf_1(t)\right)(dt)^\alpha e_1+\int_0^\infty E_\alpha(-s_2^\alpha t^\alpha)\left(kf_2(t)\right)(dt)^\alpha e_2 \\
&= k\int_0^\infty E_\alpha(-s_1^\alpha t^\alpha)f_1(t)(dt)^\alpha e_1+k\int_0^\infty E_\alpha(-s_2^\alpha t^\alpha)f_2(t)(dt)^\alpha e_2 \\
&= k\int_0^\infty E_\alpha(-\xi^\alpha t^\alpha)f(t)(dt)^\alpha \\
&= kL_\alpha\left(f(t)\right) \\
&= kF_\alpha(\xi).
\end{aligned} \tag{3.8}$$

□

Theorem 3.5 (Bicomplex Fractional Laplace Transform of Derivatives)**.** *Let $F_\alpha(\xi)$ be the bicomplex fractional LT of order α of function $f(t) \in \mathcal{C}$ and $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ 0 < \alpha < 1$ then*

$$L_\alpha\left({}^J D^\alpha f(t)\right) = \xi^\alpha F_\alpha(\xi) - f(0), \tag{3.9}$$

where ${}^J D^\alpha$ is defined in the definition (2.3).

Proof. Let $\xi = s_1e_1 + s_2e_2 \in \mathbb{T}$ and $0 < \alpha < 1$ then

$$\begin{aligned} L_\alpha\left({}^J D^\alpha f(t)\right) &= \int_0^\infty E_\alpha(-\xi^\alpha t^\alpha)\left(D^\alpha f(t)\right)(dt)^\alpha \\ &= \left[f(t)E_\alpha(-\xi^\alpha t^\alpha)\right]_0^\infty - \int_0^\infty f(t)\left(-\xi^\alpha E_\alpha(-\xi^\alpha t^\alpha)\right)(dt)^\alpha \\ &= -f(0) + \xi^\alpha \int_0^\infty f(t)E_\alpha(-\xi^\alpha t^\alpha)(dt)^\alpha \\ &= \xi^\alpha L_\alpha\left(f(t)\right) - f(0) \\ &= \xi^\alpha F_\alpha(\xi) - f(0). \end{aligned} \tag{3.10}$$

□

Corollary 3.6. *Let $F_\alpha(\xi)$ be the bicomplex fractional LT of order α of function $f(t) \in \mathcal{C}$ and $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ 0 < \alpha < 1$ then*

$$L_\alpha\left({}^J D^{2\alpha} f(t)\right) = \xi^{2\alpha} F_\alpha(\xi) - \xi^\alpha f(0) - f^\alpha(0), \tag{3.11}$$

where ${}^J D^{2\alpha}$ is defined in the definition (2.3).

Proof. Let $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ 0 < \alpha < 1$ and $D^\alpha f(t) = F(t)$ then

$$\begin{aligned} L_\alpha\left({}^J D^{2\alpha} f(t)\right) &= L_\alpha\left({}^J D^\alpha F(t)\right) \\ &= \xi^\alpha L_\alpha\left(F(t)\right) - F(0) \\ &= \xi^\alpha L_\alpha\left({}^J D^\alpha f(t))\right) - f^\alpha(0) \\ &= \xi^\alpha\left(\xi^\alpha L_\alpha\left(f(t)\right) - f(0))\right) - f^\alpha(0) \\ &= \xi^{2\alpha} L_\alpha\left(f(t)\right) - \xi^\alpha f(0) - f^\alpha(0) \\ &= \xi^{2\alpha} F_\alpha(\xi) - \xi^\alpha f(0) - f^\alpha(0). \end{aligned} \tag{3.12}$$

□

Proceeding in similar manner, we obtain the result contained in the following corollary:

Corollary 3.7. *Let $F_\alpha(\xi)$ be the bicomplex fractional LT of order α of function $f(t) \in \mathcal{C}$ and $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ 0 < \alpha < 1$ then*

$$\begin{aligned} &L_\alpha\left({}^J D^{n\alpha} f(t)\right) \\ &= \xi^{n\alpha} F_\alpha(\xi) - \left(\xi^{n\alpha-\alpha} f(0) + \xi^{n\alpha-2\alpha} f^\alpha(0) + \xi^{n\alpha-3\alpha} f^{2\alpha}(0) + \cdots + f^{n\alpha-\alpha}(0)\right), \end{aligned} \tag{3.13}$$

where ${}^J D^{n\alpha}$ is defined in the definition (2.3).

Theorem 3.8 (Change of Scale Property)**.** *Let $F_\alpha(\xi)$ be the bicomplex fractional Laplace transform of order α of function $f(t) \in \mathcal{C}$ and $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ s_1,\ s_2 \in \mathbb{C},\ a > 0$ and $0 < \alpha < 1$ then*

$$L_\alpha(f(at)) = (1/a)^\alpha F_\alpha\left(\frac{\xi}{a}\right). \tag{3.14}$$

Proof. Let $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ s_1,\ s_2 \in \mathbb{C}$ and $0 < \alpha < 1$ then from equations (3.4) and (2.14) we have

$$\begin{aligned}
L_\alpha(f(at)) &= \int_0^\infty E_\alpha(-\xi^\alpha t^\alpha) f(at)(dt)^\alpha \\
&= \lim_{\mathcal{M}\to\infty} \int_0^{\mathcal{M}} E_\alpha(-\xi^\alpha t^\alpha) f(at)(dt)^\alpha \\
&= \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}} (\mathcal{M}-t)^{\alpha-1} E_\alpha(-\xi^\alpha t^\alpha) f(at)(dt) \\
&= \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}} (\mathcal{M}-t)^{\alpha-1} E_\alpha(-s_1^\alpha t^\alpha) f_1(at)(dt)e_1 \\
&\quad + \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}} (\mathcal{M}-t)^{\alpha-1} E_\alpha(-s_2^\alpha t^\alpha) f_2(at)(dt)e_2,
\end{aligned} \tag{3.15}$$

putting $at = x \Longrightarrow dt = \frac{dx}{a},\ a > 0,$

$$\begin{aligned}
L_\alpha(f(at)) &= \lim_{\mathcal{M}\to\infty} \alpha \int_0^{a\mathcal{M}} \left(\mathcal{M} - \frac{x}{a}\right)^{\alpha-1} E_\alpha\left(-s_1^\alpha \frac{x^\alpha}{a^\alpha}\right) f_1(x)\frac{dx}{a} e_1 \\
&\quad + \lim_{\mathcal{M}\to\infty} \alpha \int_0^{a\mathcal{M}} \left(\mathcal{M} - \frac{x}{a}\right)^{\alpha-1} E_\alpha\left(-s_2^\alpha \frac{x^\alpha}{a^\alpha}\right) f_2(x)\frac{dx}{a} e_2 \\
&= \lim_{\mathcal{M}\to\infty} \alpha \int_0^{a\mathcal{M}} \frac{(a\mathcal{M}-x)^{\alpha-1}}{a^{\alpha-1}} E_\alpha\left(-s_1^\alpha \frac{x^\alpha}{a^\alpha}\right) f_1(x)\frac{dx}{a} e_1 \\
&\quad + \lim_{\mathcal{M}\to\infty} \alpha \int_0^{a\mathcal{M}} \frac{(a\mathcal{M}-x)^{\alpha-1}}{a^{\alpha-1}} E_\alpha\left(-s_2^\alpha \frac{x^\alpha}{a^\alpha}\right) f_2(x)\frac{dx}{a} e_1 \\
&= (1/a)^\alpha F_\alpha\left(\frac{\xi_1}{a}\right) e_1 + (1/a)^\alpha F_\alpha\left(\frac{\xi_2}{a}\right) e_2 \\
&= (1/a)^\alpha F_\alpha\left(\frac{\xi}{a}\right).
\end{aligned} \tag{3.16}$$

□

Theorem 3.9 (Shifting Property)**.** *Let $F_\alpha(\xi)$ be the bicomplex fractional Laplace transform of order α of $f(t) \in \mathcal{C}$ and $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ s_1,\ s_2 \in \mathbb{C},\ c > 0$ and $0 < \alpha < 1$ then*

$$L_\alpha(f(t-c)) = E_\alpha(\xi^\alpha c^\alpha) F_\alpha(\xi). \tag{3.17}$$

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Proof. Let $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ s_1,\ s_2 \in \mathbb{C}$ and $0 < \alpha < 1$ then from equations (3.4) and (2.14) we have

$$\begin{aligned}
L_\alpha(f(t-c)) &= \int_0^\infty E_\alpha(-\xi^\alpha t^\alpha) f(t-c)(dt)^\alpha \\
&= \lim_{\mathcal{M}\to\infty} \int_0^{\mathcal{M}} E_\alpha(-\xi^\alpha t^\alpha) f(t-c)(dt)^\alpha \\
&= \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}} (\mathcal{M}-t)^{\alpha-1} E_\alpha(-\xi^\alpha t^\alpha) f(t-c)(dt) \qquad (3.18)\\
&= \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}} (\mathcal{M}-t)^{\alpha-1} E_\alpha(-s_1^\alpha t^\alpha) f_1(t-c)(dt)e_1 \\
&\quad + \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}} (\mathcal{M}-t)^{\alpha-1} E_\alpha(-s_2^\alpha t^\alpha) f_2(t-c)(dt)e_2.
\end{aligned}$$

Putting $t - c = x \Longrightarrow dt = dx$

$$\begin{aligned}
&= \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}-c} (\mathcal{M}-x-c)^{\alpha-1} E_\alpha(-s_1^\alpha (x+c)^\alpha) f_1(x)(dx)e_1 \\
&\quad + \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}-c} (\mathcal{M}-x-c)^{\alpha-1} E_\alpha(-s_2^\alpha (x+c)^\alpha) f_2(x)(dx)e_2 \\
&= E_\alpha(-s_1^\alpha c^\alpha) \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}-c} (\mathcal{M}-x-c)^{\alpha-1} E_\alpha(-s_1^\alpha x^\alpha) f_1(x)(dx)e_1 \qquad (3.19)\\
&\quad + E_\alpha(-s_2^\alpha c^\alpha) \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}-c} (\mathcal{M}-x-c)^{\alpha-1} E_\alpha(-s_2^\alpha x^\alpha) f_2(x)(dx)e_2 \\
&= E_\alpha(\xi^\alpha c^\alpha) L_\alpha(f(t)) \\
&= E_\alpha(\xi^\alpha c^\alpha) F_\alpha(\xi).
\end{aligned}$$

□

Theorem 3.10 (Bicomplex Fractional Laplace Transform of Integrals)**.** *Let $F_\alpha(\xi)$ be the bicomplex fractional Laplace transform of order α of $f(t) \in \mathcal{C}$ and $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ 0 < \alpha < 1$ then*

$$L_\alpha\left(\int_0^t f(v)(dv)^\alpha\right) = \frac{1}{\xi^\alpha \Gamma(1+\alpha)} L_\alpha(f(t)). \qquad (3.20)$$

Proof. Since

$${}^J D_t^\alpha \int_0^t f(v)(dv)^\alpha = \alpha!\, f(t), \qquad (3.21)$$

by using equation (3.9)

$$\begin{aligned}
L_\alpha\left({}^J D_t^\alpha \int_0^t f(v)(dv)^\alpha\right) &= \xi^\alpha L_\alpha\left(\int_0^t f(v)(dv)^\alpha\right), \\
L_\alpha(\alpha!\, f(t)) &= \xi^\alpha L_\alpha\left(\int_0^t f(v)(dv)^\alpha\right).
\end{aligned} \qquad (3.22)$$

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Hence,

$$L_\alpha\left(\int_0^t f(v)(dv)^\alpha\right) = \Gamma(\alpha+1)\xi^{-\alpha}L_\alpha\left(f(t)\right). \tag{3.23}$$

□

Theorem 3.11. *Let $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ s_1,\ s_2 \in \mathbb{C}$ and $0<\alpha<1$ then*

(i) $L_\alpha(t^\alpha f(t)) = -{}^J D_\xi^\alpha L_\alpha(f(t))$,

(ii) $L_\alpha(E_\alpha(-c^\alpha t^\alpha)f(t))_\xi = F_\alpha(\xi+c)$,

(iii) $L_\alpha(-t^\alpha f(t)) = {}^J D_\xi^\alpha L_\alpha(f(t))$.

Proof. (i) Let $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ s_1,\ s_2 \in \mathbb{C}$ and $0<\alpha<1$ then from equation (3.4) we have

$$\begin{aligned}
&L_\alpha(t^\alpha f(t))\\
&\quad= \int_0^\infty E_\alpha(-\xi^\alpha t^\alpha)t^\alpha f(t)(dt)^\alpha\\
&\quad= \int_0^\infty E_\alpha(-s_1^\alpha t^\alpha)t^\alpha f_1(t)(dt)^\alpha e_1 + \int_0^\infty E_\alpha(-s_2^\alpha t^\alpha)t^\alpha f_2(t)(dt)^\alpha e_2\\
&\quad= -\ {}^J D_{s_1}^\alpha \int_0^\infty E_\alpha(-s_1^\alpha t^\alpha)f_1(t)(dt)^\alpha e_1 -\ {}^J D_{s_2}^\alpha \int_0^\infty E_\alpha(-s_2^\alpha t^\alpha)f_2(t)(dt)^\alpha e_2\\
&\quad= -\ {}^J D_\xi^\alpha L_\alpha(f(t)).
\end{aligned} \tag{3.24}$$

(ii) Let $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ s_1,\ s_2 \in \mathbb{C}$ and $0<\alpha<1$ then from equation (3.4) we have

$$\begin{aligned}
L_\alpha(E_\alpha(-c^\alpha t^\alpha)f(t))_\xi &= \int_0^\infty E_\alpha(-\xi^\alpha t^\alpha)E_\alpha(-c^\alpha t^\alpha)f(t)(dt)^\alpha\\
&= \int_0^\infty E_\alpha(-(\xi+c)^\alpha t^\alpha)f(t)(dt)^\alpha\\
&= F_\alpha(\xi+c).
\end{aligned} \tag{3.25}$$

(iii) Let $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ s_1,\ s_2 \in \mathbb{C}$ and $0<\alpha<1$ then from equation (3.4) we have

$$\begin{aligned}
&L_\alpha(-t^\alpha f(t))\\
&\quad= \int_0^\infty E_\alpha(-\xi^\alpha(-t)^\alpha)t^\alpha f(t)(dt)^\alpha\\
&\quad= \int_0^\infty E_\alpha(-s_1^\alpha t^\alpha)t^\alpha f_1(t)(dt)^\alpha e_1 + \int_0^\infty E_\alpha(-s_2^\alpha t^\alpha)t^\alpha f_2(t)(dt)^\alpha e_2\\
&\quad= -\ {}^J D_{s_1}^\alpha \int_0^\infty E_\alpha(-s_1^\alpha t^\alpha)f_1(t)(dt)^\alpha e_1 -\ {}^J D_{s_2}^\alpha \int_0^\infty E_\alpha(-s_2^\alpha t^\alpha)f_2(t)(dt)^\alpha e_2\\
&\quad= -\ {}^J D_\alpha^\xi L_\alpha(f(t)).
\end{aligned} \tag{3.26}$$

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□

3.2 Convolution Theorem

Convolution is a mathematical operation on two functions $f,\ g$. which is useful in signal theory, image processing. Convolution of order μ of the functions $f(t),\ g(t)$ defined by Jumarie [14] given by

$$(f*g)(t)=\int_0^t f(t-v)g(v)(dv)^{\mu}. \tag{3.27}$$

Theorem 3.12. *Let $f,\ g\in\mathcal{C}$ and Let $F_\alpha(\xi)$ and $G_\alpha(\xi)$ be the bicomplex fractional Laplace transform of order α of functions $f(t)$ and $g(t)$ respectively, then*

$$L_\alpha(f*g)(t)=F_\alpha(\xi)G_\alpha(\xi)=L_\alpha(f(t))\,L_\alpha(g(t)). \tag{3.28}$$

Proof.

$$\begin{aligned}
L_\alpha(f(t)*g(t)) &= \int_0^\infty (dt)^\alpha E_\alpha(-\xi^\alpha t^\alpha)\int_0^t f(t-v)g(v)(dv)^\alpha\\
&= \int_0^\infty (dt)^\alpha E_\alpha(-(s_1e_1+s_2e_2)^\alpha t^\alpha)\int_0^t f(t-v)g(v)(dv)^\alpha\\
&= \left(\int_0^\infty (dt)^\alpha E_\alpha(-s_1^\alpha t^\alpha)\int_0^t f(t-v)g(v)(dv)^\alpha\right)e_1\\
&\quad+\left(\int_0^\infty (dt)^\alpha E_\alpha(-s_2^\alpha t^\alpha)\int_0^t f(t-v)g(v)(dv)^\alpha\right)e_2\\
&= \left(\int_0^\infty (dt)^\alpha E_\alpha(-s_1^\alpha (t-v)^\alpha)E_\alpha(-s_1^\alpha v^\alpha)\int_0^t f(t-v)g(v)(dv)^\alpha\right)e_1\\
&\quad+\left(\int_0^\infty (dt)^\alpha E_\alpha(-s_2^\alpha (t-v)^\alpha)E_\alpha(-s_2^\alpha v^\alpha)\int_0^t f(t-v)g(v)(dv)^\alpha\right)e_2.
\end{aligned} \tag{3.29}$$

Put $p=t-v,\ q=v$, to obtain

$$\begin{aligned}
L_\alpha(f(t)*g(t)) &= \left(\int_0^\infty\int_0^\infty (dp)^\alpha E_\alpha(-s_1^\alpha p^\alpha)E_\alpha(-s_1^\alpha q^\alpha)f(p)g(q)(dq)^\alpha\right)e_1\\
&\quad+\left(\int_0^\infty\int_0^\infty (dp)^\alpha E_\alpha(-s_2^\alpha p^\alpha)E_\alpha(-s_2^\alpha q^\alpha)f(p)g(q)(dq)^\alpha\right)e_2\\
&= \left(\int_0^\infty\int_0^\infty E_\alpha(-s_1^\alpha p^\alpha)E_\alpha(-s_1^\alpha q^\alpha)f(p)g(q)(dp)^\alpha(dq)^\alpha\right)e_1\\
&\quad+\left(\int_0^\infty\int_0^\infty E_\alpha(-s_2^\alpha p^\alpha)E_\alpha(-s_2^\alpha q^\alpha)f(p)g(q)(dp)^\alpha(dq)^\alpha\right)e_2
\end{aligned} \tag{3.30}$$

Hence,

$$\begin{aligned} L_\alpha (f(t) * g(t)) &= (F_\alpha(s_1)G_\alpha(s_1))\, e_1 + (F_\alpha(s_2)G_\alpha(s_2))\, e_2 \\ &= F_\alpha(\xi)G_\alpha(\xi) \\ &= L_\alpha(f(t))L_\alpha(g(t)). \end{aligned} \tag{3.31}$$

□

4 Bicomplex Fractional Inverse Laplace Transform

Definition 4.1. Generalized Dirac's function $\delta_\alpha(x)$ of fractional order α, $0 < \alpha < 1$ is given by (see, e.g. [14])

$$\int_{\mathbb{R}} f(x)\delta_\alpha(x)(dx)^\alpha = \alpha f(0). \tag{4.1}$$

The relation between Dirac's function and ML function is given by (see, e.g. [14]) the following result

$$\frac{\alpha}{(M_\alpha)^\alpha} \int_{-i_1\infty}^{+i_1\infty} E_\alpha(i_1(-\omega x)^\alpha)(d\omega)^\alpha = \delta_\alpha(x), \tag{4.2}$$

where M_α is the period of the complex-valued ML function defined by the relation $E_\alpha(i_1(M_\alpha)^\alpha) = 1$.

Theorem 4.2. *Let $F_\alpha(\xi)$ be the bicomplex fractional Laplace transform of order α of function $f(t) \in \mathcal{C}$ and $\xi = s_1e_1 + s_2e_2 \in \mathbb{T}$ and $0 < \alpha < 1$ then*

$$f(t) = \frac{1}{(M_\alpha)^\alpha} \int_H E_\alpha(\xi^\alpha x^\alpha)F_\alpha(\xi)(d\xi)^\alpha, \tag{4.3}$$

where H is closed contour in $\mathbb{T}$.

Proof. Let $F_\alpha(\xi)$ be the bicomplex fractional Laplace transform of bicomplex-valued function $f(t)$. Then $F_\alpha(\xi) = F_\alpha(s_1)e_1 + F_\alpha(s_2)e_2$. The inverse formula for complex fractional Laplace transform (see, e.g. [14]) are

$$\begin{aligned} f_1(t) &= \frac{1}{(M_\alpha)^\alpha} \int_{-i_1\infty}^{+i_1\infty} E_\alpha(s_1^\alpha x^\alpha)F_\alpha(s_1)(ds_1)^\alpha \\ &= \frac{1}{(M_\alpha)^\alpha} \int_{\gamma_1} E_\alpha(s_1^\alpha x^\alpha)F_\alpha(s_1)(ds_1)^\alpha, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} f_2(t) &= \frac{1}{(M_\alpha)^\alpha} \int_{-i_1\infty}^{+i_1\infty} E_\alpha(s_2^\alpha x^\alpha)F_\alpha(s_2)(ds_2)^\alpha \\ &= \frac{1}{(M_\alpha)^\alpha} \int_{\gamma_2} E_\alpha(s_2^\alpha x^\alpha)F_\alpha(s_2)(ds_2)^\alpha, \end{aligned} \tag{4.5}$$

where M_α is the period of the complex-valued ML function defined by the relation $E_\alpha(i_1(M_\alpha)^\alpha) = 1$ and γ_1 and γ_2 be closed contours taken along the the vertical lines as follows $\gamma_1 = -i_1\infty$ to $i_1\infty$,$\gamma_2 = -i_1\infty$ to $i_1\infty$.

Now, using complex inversions (4.4) and (4.5), we get

$$\begin{aligned}
f(t) &= f_1(t)e_1 + f_2(t)e_2 \\
&= \frac{1}{(M_\alpha)^\alpha}\left(\int_{\gamma_1} E_\alpha(s_1^\alpha x^\alpha)F_\alpha(s_1)(ds_1)^\alpha\ e_1 + \int_{\gamma_2} E_\alpha(s_2^\alpha x^\alpha)F_\alpha(s_2)(ds_2)^\alpha\ e_2\right) \\
&= \frac{1}{(M_\alpha)^\alpha}\int_{(\gamma_1,\gamma_2)} E_\alpha((s_1e_1+s_2e_2)^\alpha x^\alpha)F_\alpha(s_1e_1+s_2e_2)\left((ds_1)^\alpha e_1 + (ds_2)^\alpha e_2\right) \\
&= \frac{1}{(M_\alpha)^\alpha}\int_H E_\alpha(\xi^\alpha x^\alpha)F_\alpha(\xi)(d\xi)^\alpha,
\end{aligned} \tag{4.6}$$

where $H = (\gamma_1, \gamma_2)$ and

$$\xi = s_1e_1 + s_2e_2 \Rightarrow d\xi = ds_1e_1 + ds_2e_2 \Rightarrow (d\xi)^\alpha = (ds_1)^\alpha e_1 + (ds_2)^\alpha e_2. \tag{4.7}$$

□

5 Application of Bicomplex Fractional Laplace Transform

Agarwal et al.[3] discussed fractional differential equations in bicomplex space. Bicomplex fractional Laplace transform has great advantage in finding the solution of fractional order differential equations. We have solved the following homogeneous fractional order differential equations using bicomplex fractional Laplace transform.

$$(D^{2\alpha} + 2D^\alpha + 2)y(t) = 0, \tag{5.1}$$

where $y(0) = 1$ and $y^\alpha(0) = -1$.
By taking bicomplex fractional LT on both sides of order α, we get

$$L_\alpha\left(y^{2\alpha} + 2y^\alpha + 2y\right) = 0, \tag{5.2}$$

$$s^{2\alpha}L_\alpha(y(t)) - s^\alpha y(0) - y^\alpha(0) + 2\left(s^\alpha L_\alpha y(t) - y(0)\right) + 2L_\alpha y(t) = 0, \tag{5.3}$$

$$\left(s^{2\alpha} + 2s^\alpha + 2\right)L_\alpha y(t) = s^\alpha + 1, \tag{5.4}$$

$$\Rightarrow \quad L_\alpha y(t) = \frac{s^\alpha + 1}{(s^\alpha+1)^2 + 1}. \tag{5.5}$$

Hence,

$$L_\alpha y(t) = L_\alpha\left(\mathbb{E}_\alpha(-t^\alpha)\cos_\alpha(t^\alpha)\right). \tag{5.6}$$

Therefore

$$y(t) = \mathbb{E}_\alpha(-t^\alpha)\cos_\alpha(t^\alpha), \tag{5.7}$$

where $\cos_\alpha(t^\alpha)$ is fractional order cosine function (see, e.g. [13, 19]).

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5.1 Application to Diffusion equation

Consider the following partial fractional differential equation

$$D_t^{\alpha} u(x,t) = c D_x^{\beta} u(x,t),\ 0 < \alpha, \beta < 1, \tag{5.8}$$

with initial condition $u(x,t) = f(x)$. It is very simple case of diffusion equation (see, e. g. [13]).
By taking bicomplex fractional LT of the equation (5.8) with respect to t,

$$s^{\alpha} \bar{u}(x,s) - f(x) = c D_x^{\beta} \bar{u}(x,s). \tag{5.9}$$

Taking fractional Fourier transform of equation (5.9) defined by Jumarie [13] with respect to x,

$$s^{\alpha} \hat{\bar{u}}(\zeta,s) - \hat{f}(\zeta) = c(-i_1 \zeta^{\beta}) \hat{\bar{u}}(\zeta,s), \tag{5.10}$$

or

$$(s^{\alpha} + i_1 c \zeta^{\beta}) \hat{\bar{u}}(\zeta,s) = \hat{f}(\zeta), \tag{5.11}$$

$$\hat{\bar{u}}(\zeta,s) = \frac{\hat{f}(\zeta)}{(s^{\alpha} + i_1 c \zeta^{\beta})}. \tag{5.12}$$

By taking inverse Bicomplex fractional Laplace transform

$$\hat{u}(\zeta,t) = \hat{f}(\zeta) E_{\alpha}(-i_1 c \zeta^{\beta} t^{\alpha}), \qquad \text{(From [19, Property 3.4])}. \tag{5.13}$$

Finally by taking Inverse Fractional Fourier transform defined by Jumarie [13] of the equation (5.13)

$$u(x,t) = \frac{1}{(M_{\beta})^{\beta}} \int_{-\infty}^{+\infty} E_{\beta}(i_1 \zeta^{\beta} x^{\beta})\ E_{\alpha}(-i_1 c \zeta^{\beta} t^{\alpha})\ \hat{f}(\zeta)\ (d\zeta)^{\alpha}. \tag{5.14}$$

6 Conclusion

In this paper, the Laplace transform of fractional order or fractional Laplace transform in bicomplex space, the extension of complex Laplace transform of fractional order has been derived. Various properties along with the convolution theorem have also been derived. Bicomplex fractional Laplace transform may be used in finding the solution of bicomplex fractional Schrödinger equation.

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